

GENUS ONE, THREE-BRIDGE KNOTS ARE PRETZEL

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ABSTRACT. We prove that genus one, three-bridge knots are pretzel knots.

1. INTRODUCTION

In knot theory, there are some evidences to specialize genus one knots. For instance, the trefoil and the figure-eight knot are the only fibered knots of genus one. Genus one, unknotting number one knots are doubled knots [2, 3].

In this note we will give a new feature of genus one knots.

Theorem 1.1. *Genus one, three-bridge knots are pretzel knots.*

It is well known that genus one, two-bridge knots are $C[2b, 2c]$ ($b, c \in \mathbb{Z}$) in Conway's notation. It seems to be impossible to give a concise description, such as Theorem 1.1, for genus one knots with bridge index more than three.

The proof of Theorem 1.1 will be done mainly by analyzing the intersections of a minimal genus Seifert surface of a knot and a certain level sphere.

2. PROOF OF THEOREM

2.1. For an arc α properly embedded in a surface F , α is said to be *inessential* if there is a subarc β of ∂F such that $\alpha \cup \beta$ bounds a disk in F , and *essential*, otherwise.

Let K be a knot in the 3-sphere S^3 . Recall the definition of a thin position of a knot [1]. Let $\{\pm\infty\}$ be the north and south poles of S^3 . Then $S^3 - \{\pm\infty\}$ is naturally homeomorphic to $S^2 \times R^1$. Let $h : S^3 - \{\pm\infty\} \rightarrow R^1$ be the projection. For $t \in R^1$, $h^{-1}(t)$ is called a level sphere. By an isotopy of K , we may assume that $K \subset S^3 - \{\pm\infty\}$ and that $h|_K$ is a Morse function. In particular, all critical values are distinct.

Let Q_1, \dots, Q_m be level spheres, one between each consecutive pair of critical points of $h|_K$. Then $\sum_{i=1}^m |S_i \cap K|$ is called the complexity of the Morse position. A *thin position* of K is a Morse position of minimal complexity.

Lemma 2.1. *Suppose that K is a prime 3-bridge knot. Then a 3-bridge position of K is a thin position of K .*

Proof. Assume that K is in a 3-bridge position. Then its complexity is 18. All possible complexities of a knot less than 18 are 2, 8 and 14. However, these complexities are realized by only the unknot, 2-bridge knots, and connected sums of two 2-bridge knots, respectively. Therefore 18 is minimal for K . \square

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Hereafter, we assume that K is genus one, 3-bridge knot. Let S be a genus one Seifert surface of K .

Lemma 2.2. *There exists a 2-sphere Q giving a 3-bridge position of K such that Q is transverse to S and each arc component of $S \cap Q$ is essential in S .*

Proof. It follows from Lemma 2.1 and [1, Lemma 4.4]. Remark that such Q is chosen as a level sphere between the highest local minimum and the lowest local maximum of K by the proof of [1, Lemma 4.4]. \square

Then $S \cap Q$ consists of three arcs and possibly some loops. We assume that $|S \cap Q|$ is minimal up to an isotopy of S . Since each arc component of $S \cap Q$ is essential in S , there are three possibilities:

- (i) there is no parallel arcs in S ;
- (ii) three arcs are mutually parallel in S ;
- (iii) only two of them are mutually parallel in S .

It is easy to see that the last case is impossible, since Q is separating in S^3 . Let B_1 and B_2 be the 3-balls bounded by Q . Then $(B_i, B_i \cap K)$ is a 3-string trivial tangle for $i = 1, 2$.

Lemma 2.3. *There is no circle components of $S \cap Q$ bounding a disk in either S or $Q - K$.*

Proof. By the incompressibility and minimality of S , there is no circle component of $S \cap Q$ bounding a disk in $Q - K$.

Let ξ be a circle component of $S \cap Q$ which bounds a disk D in S . It may be assumed that $\text{Int}D \cap Q = \emptyset$, and $D \subset B_1$. Then D separates three strings $B_1 \cap K$ in B_1 . Let D' be the disk in Q bounded by ξ such that D' meets K in two points. Then $D \cup D'$ bounds a 3-ball $B \subset B_1$, which contains exactly one string t of $B_1 \cap K$. Note that $D' \cap S$ consists of exactly one arc e , connecting two endpoints of t , and loops. Then endpoints of e are consecutive on ∂S , and therefore the possible configuration is Case (ii).

Let S' be the component of $S \cap B$ meeting t . Then it is a punctured annulus. Let ζ be the boundary component of S' which is disjoint from t and essential in S . Note that ζ is essential in S . Then ζ bounds a disk E in $D \cup (D' - e)$. By pushing E slightly into $\text{Int}B$, we may assume that E meet $S \cap B$ transversely. If $\text{Int}E \cap S = \emptyset$, then E gives a compressing disk for S . Otherwise, it is easy to give a compressing disk for S by a cut-and-paste argument. \square

Thus, in Case (i), $S \cap Q$ does not contain circle components, and in Case (ii) it contains an odd number of circle components by the fact that Q is separating in S^3 .

The remainder of this note will be devoted to prove that each case leads to the desirable conclusion that K is pretzel.

2.2. Case (i). In this case, $B_i \cap S$ consists of a disk for $i = 1, 2$. We will examine the situation in B_1 , but it is similar to that in B_2 .

Consider the disk component $f = S \cap B_1$. Then ∂f consists of an alternating sequence of three arcs e_1, e_2, e_3 of $S \cap Q$ and the strings t_1, t_2, t_3 of $K \cap B_1$. By Lemma 2.3, $\text{Int}f \cap Q = \emptyset$.

Lemma 2.4. *There exist mutually disjoint disks D_i such that ∂D_i is the union of t_i and an arc in ∂B_1 and $D_i \cap f = t_i$ for $i = 1, 2, 3$.*

Proof. Since $(B_1, B_1 \cap K)$ is a trivial tangle, there is a disk D_1 such that ∂D_1 is the union of t_1 and an arc in ∂B_1 .

Suppose that $\text{Int}D_1 \cap f \neq \emptyset$. Then we may assume that $\text{Int}D_1 \cap f$ consists of loops and arcs. By a cut-and-paste argument, we may also assume that $\text{Int}D_1 \cap f$ contains no loop components. Furthermore, it can be assumed that each arc component of $\text{Int}D_1 \cap f$ has its endpoints on ∂B_1 , by an isotopy of D_1 .

On f , each arc of $\text{Int}D_1 \cap f$ has its endpoints either on some e_j , or on e_k and e_l for $k \neq l$. The former one can be removed by cut-and-paste of D_1 . Therefore we may assume that $\text{Int}D_1 \cap f$ contains only the latter type of arcs. Choose an outermost one ξ on f . Then there is a rectangle R in f cut off from f by ξ . Let D' be the disk cut off from D_1 by ξ such that D' is disjoint from t_1 . Then we replace D_1 by $D'_1 = R \cup D'$. By pushing D'_1 away from f , we obtain a new disk D'_1 such that $|D'_1 \cap f| < |D_1 \cap f|$. Note that if $t_j \subset R$ then the new D'_1 meets the string t_j , not necessarily t_1 .

Thus we may assume that there is a disk D_1 such that ∂D_1 is the union of t_1 and an arc in ∂B_1 , and that $D_1 \cap f = t_1$.

Let $F = f \cup D_1$. Then F is a disk in B_1 giving a parallelism between the remaining two strings t_2 and t_3 . Since $(B_1, B_1 \cap K)$ is trivial, F is unknotted, and therefore it is obvious that there are two disjoint disks D_2 and D_3 such that ∂D_j is the union of t_j and an arc in ∂B_1 and that $D_j \cap F = t_j$, for $j = 2, 3$. \square

By Lemma 2.4, there is a disk E_1 properly embedded in B_1 , such that $f \subset E_1$. Similarly, g is contained in a properly embedded disk $E_2 \subset B_2$. Then S is restored by gluing f and g along three arcs of $S \cap Q$. Thus it is clear to see that K is a pretzel knot with three strands each having an odd number of half twists.

2.3. Case (ii). Recall that $S \cap Q$ consists of three arcs e_1, e_2, e_3 and an odd number of loop components, since Q is separating in S^3 , and that each loop component of $S \cap Q$ is essential in both S and $Q - K$ by Lemma 2.3.

Let $S_i = S \cap B_i$ for $i = 1, 2$. Then S_i consists of a disk b_i , giving a parallelism between two strings in B_i , and annuli. In particular, S_i contains the unique annulus component c_i meeting the remaining string in B_i . Let $c_i \cap Q = e_i \cup \gamma_i$, where γ_i is a loop. See Figure 1.

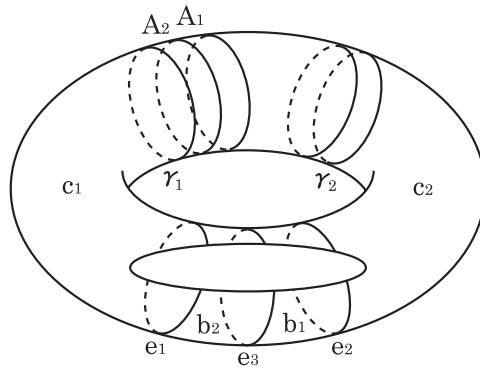


FIGURE 1

Since $(B_i, B_i \cap K)$ is trivial, there exists a disk D_i properly embedded in B_i which separates b_i and the remaining string of $B_i \cap K$ disjoint from b_i . We can assume that $|D_i \cap S_i|$ is minimal up to an isotopy of D_i , and that $D_i \cap S_i$ consists of essential arcs in $S_i - K$.

Lemma 2.5. *The loop γ_i separates two arcs $b_i \cap Q$ in Q .*

Proof. Let α be an outermost arc of $D_i \cap S_i$ in D_i , and let δ be the outermost disk cut off from D_i by α . Let $\beta = \text{cl}(\partial\delta - \alpha)$. There are three possibilities of α .

- (1) α is an essential arc in some annulus component A of S_i ;
- (2) α is an inessential arc in c_i , but an essential arc in $c_i - K$.
- (3) α is an essential arc in c_i .

For (1), A separates B_i into a 3-ball W and a solid torus V containing δ . (We remark that the existence of δ implies that A is unknotted.)

If $\partial V - A$ does not meet K , then an isotopy of A along V reduces $|S \cap Q|$, and this contradicts the minimality of $|S \cap Q|$.

If $\partial V - A$ meets K , then W contains b_i , since each loop component of $S \cap Q$ is essential in $Q - K$, and therefore V contains c_i .

Let Δ be a disk obtained by a boundary-compression of A along δ . Then Δ cuts a 3-ball off from B_i , which contains the annulus c_i . It is easy to obtain a compressing disk for c_i and hence for S . Therefore (1) is impossible.

For (2), if β is isotopic to a subarc of e_i in $Q - K$ (hence in $Q - S$), then we would have a compressing disk for c_i , by sliding β onto the subarc of e_i . Therefore, γ_i must separate two arcs $b_i \cap Q$ in Q .

For (3), let Q_1 and Q_2 be disks bounded by γ_i in Q . Suppose that two arcs $b_i \cap Q$ are contained in Q_1 , say. Then the arcs e_i and β lie in Q_2 , and therefore there is no loop component of $S \cap Q$ in Q_2 . Since $N(c_i \cup \delta \cup Q_2; B_i)$ is a 3-ball which contains c_i only among the components of S_i , c_i is compressible there. This contradicts the incompressibility of S . \square

Lemma 2.6. $\gamma_1 = \gamma_2$.

Proof. Suppose not. Then S_i contains at least one annulus component disjoint from K . Let A be such an annulus component of S_i . First, we show that A separates b_i and c_i in B_i . Let W be the 3-ball cut off from B_i by A , and let $V = \text{cl}(B_i - W)$.

Suppose that A does not separate b_i and c_i in B_i . Then both b_i and c_i are contained in W . If V is not a solid torus, then it is a non-trivial knot exterior, and in particular, $\pi_1(V)$ is not free. Since A is incompressible, there is an injection $\pi_1(V) \rightarrow \pi_1(B_i - K)$. This contradicts that the latter is free. Therefore V is a solid torus, and there is an isotopy of A along V reducing $|S \cap Q|$, a contradiction. Therefore such an annulus A separates b_i and c_i in B_i . Furthermore, b_i is contained in W , and c_i is contained in V .

Let A_i be an annulus component of S_i disjoint from K for $i = 1, 2$ such that $c_1 \cap A_2 = \gamma_1$ and that A_1 and A_2 are adjacent on S . Let W_i be the 3-ball cut off from B_i by A_i , and let V_i be the solid torus $\text{cl}(B_i - W_i)$.

Since $c_1 \subset V_1$, $e_1 \subset V_1$. Then e_1 lies on ∂W_2 , and therefore e_2 must lie on ∂W_2 since $e_2 \subset b_1 \subset W_1$. This contradicts that c_2 is contained in V_2 . \square

Lemma 2.7. *There is a disk d_i in B_i for $i = 1, 2$ such that $b_i \cap d_i = \emptyset$, $c_i \cap d_i = K \cap c_i$, and $c_i \cup d_i$ is an annulus properly embedded in B_i .*

Proof. Let δ be an outermost disk which was used in the proof of Lemma 2.5. Recall that (1) never occur. For (2), let δ' be the disk cut off from c_i by α . Then $\delta \cup \delta'$ gives a desired disk d_i after a suitable isotopy.

For (3), let d_i be a disk obtained by a boundary-compression of c_i along δ . After an isotopy of d_i , we have a desired disk. \square

We note that the annulus $c_i \cup d_i$ is unknotted in B_i , since the band b_i is unknotted in B_i . Let Q_0 be a disk in Q which is bounded by $\gamma_1 (= \gamma_2)$ and meets b_1 . Take a thin neighborhood N of Q_0 and put $M = \text{cl}(S^3 - N)$. Then by Lemma 2.7, we may assume that $(M, M \cap K)$ is a Montesinos tangle of type $(\pm 1/3, \mp 1/3)$ and that $(N, N \cap K)$ is a rational tangle whose slope is an odd integer n . See Figure 2.

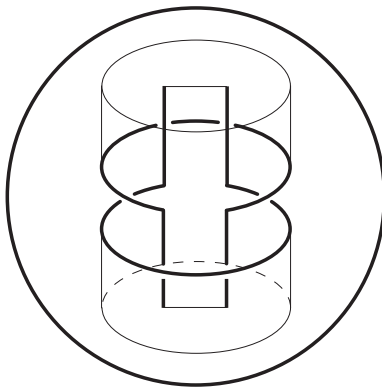


FIGURE 2

Hence K is a pretzel knot of type $(\pm 3, \mp 3, n)$. Here, $|n| > 1$, or K would be 2-bridge.

This completes the proof of Theorem 1.1. \square

Remark 2.8. Of course, a pretzel knot of type $(\pm 3, \mp 3, n)$ (n is odd) has a minimal genus Seifert surface as in Case(i) as well. However, it is not hard to deform a minimal genus Seifert surface as in Case (ii) to one as in Case (i) by an isotopy in S^3 . Therefore, the minimal genus Seifert surfaces for genus one, 3-bridge knots are unique.

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