Positive recurrence of multidimensional reflecting random walk with a background process

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Contents

d-dimensional RRW-WBP: Definition

Notations

- $N = \{1, 2, ..., d\}$
- $S^A = \{1, 2, ..., s^A\}$, $A \subset N$: finite sets, $\quad s^A$: a positive integer
- $\varphi(\boldsymbol{x}) = \{l \in N : x_l \geq 1\}$: the index set of nonzero elements in $\boldsymbol{x} \in \mathbb{Z}_+^d$

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d-dimensional skip-free reflecting random walk with a background process (dD-RRW-WBP)

$\{\boldsymbol{Y}_n\} = \{(\boldsymbol{X}_n, J_n)\}$ is a d-dimensional skip-free RRW-WBP if:

- $\{\boldsymbol{Y}_n\}$ is a Markov chain on state space $\mathcal{S} = \bigcup_{\boldsymbol{x} \in \mathbb{Z}_+^d} \left(\{\boldsymbol{x}\} \times S^{\varphi(\boldsymbol{x})}\right)$.
- $\{\boldsymbol{X}_n\} = \{(X_{1,n}, X_{2,n}, \cdots, X_{d,n})\}$ on $\mathbb{Z}_+^d$ is skip-free in all directions.
- Transition probabilities have the following space-homogeneity property:

  For $\boldsymbol{x} \in \mathbb{Z}_+^d$ and $\boldsymbol{z} \in \{-1, 0, 1\}^d$ such that $\boldsymbol{x} + \boldsymbol{z} \in \mathbb{Z}_+^d$,

$$P\big(\boldsymbol{Y}_{n+1} = (\boldsymbol{x} + \boldsymbol{z}, j) \,|\, \boldsymbol{Y}_n = (\boldsymbol{x}, i)\big) = p_{\boldsymbol{z}}^{\varphi(\boldsymbol{x})\, \varphi(\boldsymbol{x}+\boldsymbol{z})}(i, j).$$

Transition probabilities

- $\mathcal{B}^A$, $A \subset N$: Boundary faces

$$\begin{aligned} \mathcal{B}^A &= \{(\boldsymbol{x}, i) \in \mathcal{S} : \varphi(\boldsymbol{x}) = A\} \\ &= \{(\boldsymbol{x}, i) \in \mathcal{S} : \text{if } l \in A \text{ then } x_l \geq 1;\ \text{otherwise } x_l = 0\} \end{aligned}$$

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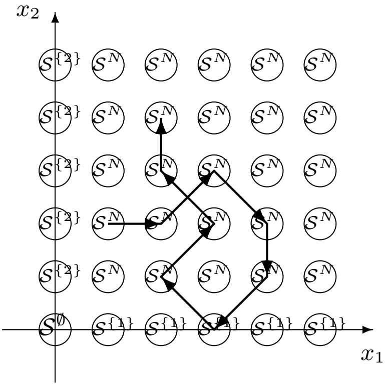
- Transition probabilities are space-homogeneous in each boundary face $\mathcal{B}^A$.

Two dimensional case

Case of $d = 2$

$\{\boldsymbol{Y}_n\} = \{(\boldsymbol{X}_n, J_n)\} = \{((X_{1,n}, X_{2,n}), J_n\}$: 2D-RRW-WBP on state space $\mathcal{S}$



$N = \{1, 2\}$

$$\varphi(\boldsymbol{x}) = \begin{cases} \emptyset & \text{if } x_1 = 0 \text{ and } x_2 = 0 \\ \{1\} & \text{if } x_1 > 0 \text{ and } x_2 = 0 \\ \{2\} & \text{if } x_1 = 0 \text{ and } x_2 > 0 \\ N & \text{otherwise} \end{cases}$$

Boundary faces:

$$\mathcal{B}^{\emptyset} = \{0\} \times \{0\} \times \mathcal{S}^{\emptyset}, \quad \mathcal{B}^{\{1\}} = \{0\} \times \mathbb{N} \times \mathcal{S}^{\{1\}}$$
$$\mathcal{B}^{\{2\}} = \mathbb{N} \times \{0\} \times \mathcal{S}^{\{2\}}, \quad \mathcal{B}^{N} = \mathbb{N} \times \mathbb{N} \times \mathcal{S}^{N}$$

State space: $\mathcal{S} = \mathcal{B}^{\emptyset} \cup \mathcal{B}^{\{1\}} \cup \mathcal{B}^{\{2\}} \cup \mathcal{B}^{N}$

- $\{\boldsymbol{Y}_n\}$ is space-homogenous in each boundary face.

An Example

Discrete-time version of d-node generalized Jackson network with Markovian arrivals and phase-type services

- $\boldsymbol{Y}_n = ((X_{1,n}, X_{2,n}, \cdots, X_{d,n}), J_n)$: the state of the network at time n
- $X_{l,n}$: the number of customers in node l
- J_n: the phase of the process combining the Markovan arrival processes and phase-type service processes

Assumption and Conditional mean increment vector

Assumption

We assume the d-dimensional RRW-WBP $\{\boldsymbol{Y}_n\}$ is irreducible.

Assumption and Conditional mean increment vector

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Conditional mean increment vector

For $\boldsymbol{y} \in \mathcal{S}$, let $\boldsymbol{\alpha}(\boldsymbol{y})$ be the conditional mean increment vector defined as

$$\boldsymbol{\alpha}(\boldsymbol{y}) = \begin{pmatrix} \alpha_1(\boldsymbol{y}) & \alpha_2(\boldsymbol{y}) & \cdots & \alpha_d(\boldsymbol{y}) \end{pmatrix} = E(\boldsymbol{\xi}_{n+1} \,|\, \boldsymbol{Y}_n = \boldsymbol{y}),$$

where $\boldsymbol{\xi}_{n+1} = \boldsymbol{X}_{n+1} - \boldsymbol{X}_n$.

Our Aim

- *We obtain sufficient conditions on which the d-dimensional RRW-WBP is positive recurrent or transient.*

Contents

1. d-Dimensional RRW with a Background Process
2. Induced Chain
3. Results
4. An Queueing Example (3D Case)

Induced chain: Definition

Notation:

- For $\boldsymbol{x} \in \mathbb{Z}_+^d$ and $A \subset N$, $\boldsymbol{x}^A = (x_l,\, l \in A)$.
- Example: $d = 5$, $N = \{1,2,3,4,5\}$, $A = \{1,3\}$

  $\Rightarrow \boldsymbol{x}^A = (x_1, x_3)$, $\boldsymbol{y}^{N\setminus A} = (y_2, y_4, y_5)$, $(\boldsymbol{x}^A, \boldsymbol{y}^{N\setminus A}) = (x_1, y_2, x_3, y_4, y_5)$

Induced chain: Definition

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Induced chain

For $A \subset N$, $A \neq \emptyset$, induced chain $\mathcal{L}^A = \{(\hat{\boldsymbol{X}}_n^{N\setminus A}, \hat{J}_n)\}$ is a $(d-|A|)$-dimensional RRW-WBP on state space $\mathcal{S}^A$.

- $\mathcal{S}^A = \bigcup_{\boldsymbol{x}^{N\setminus A} \in \mathbb{Z}_+^{d-|A|}} \Big(\{\boldsymbol{x}^{N\setminus A}\} \times S^{\varphi(\boldsymbol{x}^{N\setminus A}, \mathbf{1}^A)} \Big)$, $\mathbf{1} = (1\ 1\ \cdots\ 1) \in \mathbb{Z}_+^d$
- Transition probabilities; For $\boldsymbol{x}^{N\setminus A} \in \mathbb{Z}_+^{d-|A|}$ and $\boldsymbol{z}^{N\setminus A} \in \{-1,0,1\}^{d-|A|}$,

$$P\Big((\hat{\boldsymbol{X}}_{n+1}^{N\setminus A}, \hat{J}_{n+1}) = (\boldsymbol{x}^{N\setminus A} + \boldsymbol{z}^{N\setminus A}, j) \,\Big|\, (\hat{\boldsymbol{X}}_n^{N\setminus A}, \hat{J}_n) = (\boldsymbol{x}^{N\setminus A}, i) \Big)$$

$$= \sum_{\boldsymbol{z}^A \in \{-1,0,1\}^{|A|}} p_{(\boldsymbol{z}^{N\setminus A}, \boldsymbol{z}^A)}^{\varphi(\boldsymbol{x}^{N\setminus A}, \mathbf{1}^A)\, \varphi(\boldsymbol{x}^{N\setminus A} + \boldsymbol{z}^{N\setminus A}, \mathbf{1}^A)}(i,j)$$

Assumption and Mean increment vector

Assumption

For all $A \subset N,\ A \neq \emptyset$, *induced chain* $\mathcal{L}^A$ *is irreducible.*

Assumption and Mean increment vector

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For all $A \subset N,\ A \neq \emptyset$, induced chain $\mathcal{L}^A$ is irreducible.

For $A \subset N$, $A \neq \emptyset$, if $\mathcal{L}^A$ is positive recurrent, then we define follows:

Mean increment vector with respect to $\mathcal{L}^A$

- $\boldsymbol{\pi}^A = (\pi^A(\boldsymbol{x}^{N\setminus A}, i),\ (\boldsymbol{x}^{N\setminus A}, i) \in \mathcal{S}^A)$: the stationary distribution of $\mathcal{L}^A$
- $\boldsymbol{a}(A) = (a_1(A)\ a_2(A)\ \cdots\ a_d(A))$: Mean increment vector

$$a_l(A) = \sum_{(\boldsymbol{x}^{N\setminus A}, i) \in \mathcal{S}^A} \alpha_l((\boldsymbol{x}^{N\setminus A}, \mathbf{1}^A), i)\, \pi^A(\boldsymbol{x}^{N\setminus A}, i),\ l \in N,$$

($\alpha_l(\boldsymbol{y}) = E(X_{l,n+1} - X_{l,n}\,|\,\boldsymbol{Y}_n = \boldsymbol{y})$, $\boldsymbol{y} \in \mathcal{S}$: the conditional mean increments.)

Assumption and Mean increment vector

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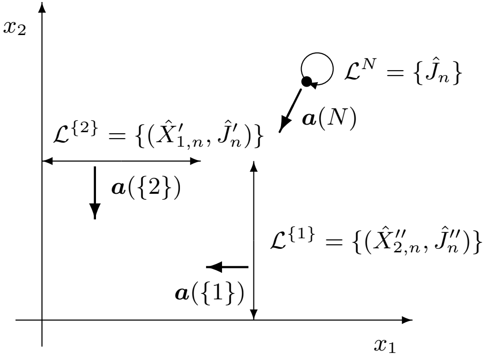
1. Note that $a_l(A) = 0$ for all $l \in N \setminus A$. Thus, $\boldsymbol{a}(A)$ is perpendicular to induced chain $\mathcal{L}^A$, in some sense.

Two dimensional case

$\{\boldsymbol{Y}_n\} = \{((X_{1,n}, X_{2,n}), J_n)\}$: 2D-RRW-WBP

Induced chains

- $\mathcal{L}^N$: finite Markov chain; $\boldsymbol{a}(N) = (a_1(N), a_2(N))$ always exists.
- $\mathcal{L}^{\{1\}}$: quasi-birth-and-death (QBD) process; positive recurrent if $a_2(N) < 0$.
- $\mathcal{L}^{\{2\}}$: QBD process; positive recurrent if $a_1(N) < 0$.



$N = \{1, 2\}$

Mean increment vectors:

$$\boldsymbol{a}(\{1\}) = \big(a_1(\{1\}),\ 0\big)$$
$$\boldsymbol{a}(\{2\}) = \big(0,\ a_2(\{2\})\big)$$
$$\boldsymbol{a}(N) = \big(a_1(N),\ a_2(N)\big)$$

Contents

A sufficient condition for positive recurrence

Notations

- $\mathcal{N}_p$: the index set of all positive-recurrent induced chains, given by

$$\mathcal{N}_p = \{A \subset N : A \neq \emptyset,\ \mathcal{L}^A \text{ is positive recurrent}\}.$$

- $\mathcal{U}$: the set of $d \times d$ matrices defined by

$$\mathcal{U} = \Big\{ U = (\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_d) :\ U \text{ is positive definite and } \boldsymbol{a}(A)^\top \boldsymbol{u}_j < 0 \text{ for all } A \in \mathcal{N}_p \text{ and all } j \in A \Big\}.$$

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Theorem (Positive recurrence)

The d-dimensional RRW-WBP is positive recurrent if $\mathcal{U} \neq \emptyset$.

- This theorem can be proved by Foster's theorem.

A sufficient condition for transience

Notations

- $\bar{\mathcal{N}}_p^A$ ($A \subset N$, $A \neq \emptyset$): the index set of positive-recurrent induced chains defined by

$$\bar{\mathcal{N}}_p^A = \bigcup_{l \in A} \Big\{ B \subset N : l \in B,\ \mathcal{L}^B \text{ is positive recurrent} \Big\}.$$

- $\mathcal{W}_A$: the set of d-dimensional vectors defined by

$$\mathcal{W}_A = \Big\{ \boldsymbol{w} = (\boldsymbol{w}^A, \boldsymbol{w}^{N \setminus A}) \in \mathbb{R}^d : \\ \boldsymbol{w}^A > \mathbf{0}^A,\ \boldsymbol{w}^{N \setminus A} < \mathbf{0}^{N \setminus A},\ \boldsymbol{a}(B)^\top \boldsymbol{w} > 0 \text{ for all } B \in \bar{\mathcal{N}}_p^A \Big\},$$

where $\mathbf{0} = (\mathbf{0}^A, \mathbf{0}^{N \setminus A})$ is a vector of 0's.

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where $\boldsymbol{0} = (\boldsymbol{0}^A, \boldsymbol{0}^{N \setminus A})$ is a vector of 0's.

Theorem (Transience)

The d-dimensional RRW-WBP is transient if $\mathcal{W}_A \neq \emptyset$ for some $A \subset N$, $A \neq \emptyset$.

Example: Two dimensional case

Classification of 2D-RRWs-WBP

C1) When $a_1(N) < 0$ and $a_2(N) < 0$, the 2D-RRW-WBP $\{\boldsymbol{Y}_n\}$ is

- (a) positive recurrent if $a_1(\{1\}) < 0$ and $a_2(\{2\}) < 0$,
- (b) transient if either $a_1(\{1\}) > 0$ or $a_2(\{2\}) > 0$.

C2) When $a_1(N) > 0$ and $a_2(N) < 0$, the 2D-RRW-WBP $\{\boldsymbol{Y}_n\}$ is

- (a) positive recurrent if $a_1(\{1\}) < 0$,
- (b) transient if $a_1(\{1\}) > 0$.

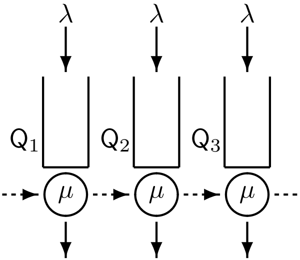
C3) When $a_1(N) < 0$ and $a_2(N) > 0$, the 2D-RRW-WBP $\{\boldsymbol{Y}_n\}$ is

- (a) positive recurrent if $a_2(\{2\}) < 0$,
- (b) transient if $a_2(\{2\}) > 0$.

C4) When $a_1(N) > 0$ and $a_2(N) > 0$, the 2D-RRW-WBP $\{\boldsymbol{Y}_n\}$ is transient.

Contents

Three queues with service interactions: A toy model



1. Three M/M/1 queues with **server vacations**, **input rejections** and **service interactions**; λ: arrival rate; μ: service rate; We assume $\rho = \lambda/\mu < 1$.
2. When a queue becomes empty, the server of the queue enters an exponentially distributed single vacation with mean $1/\delta$.

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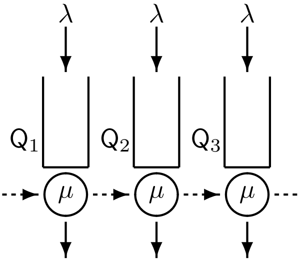

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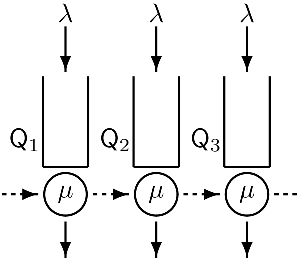

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2. When a queue becomes empty, the server of the queue enters an exponentially distributed single vacation with mean $1/\delta$.
3. Customers arriving at the queue during the vacation are rejected.
4. After the vacation, the sever enters an idle state and waits for customers.
5. When the server of Q_l enters a vacation, the server of $\mathrm{Q}_{(l+1) \bmod 3}$ begins to suspend its service; When the server of Q_l ends the vacation, the server of $\mathrm{Q}_{(l+1) \bmod 3}$ resumes its service.

Three queues with service interactions: 3D-RRW-WBP

1. This model can be represented as a continuous-time version of 3D-RRW-WBP.
2. By the uniformization, we obtain a discrete-time 3D-RRW-WBP $\{\boldsymbol{Y}_n\} = \{((X_{1,n}, X_{2,n}, X_{3,n}), J_n)\}$; $X_{l,n}$ is the number of customers in Q_l and J_n is the states of the servers. (ν: uniformization parameter)

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3. Induced chains:
   - $\mathcal{L}^N$: finite Markov chain, positive recurrent
   - $\mathcal{L}^{\{1,2\}}$, $\mathcal{L}^{\{2,3\}}$, $\mathcal{L}^{\{3,1\}}$: 1D-RRWs-WBP (QBD processes)
   - $\mathcal{L}^{\{1\}}$, $\mathcal{L}^{\{2\}}$, $\mathcal{L}^{\{3\}}$: 2D-RRWs-WBP
4. Mean increments with respect to induced chain $\mathcal{L}^N$:

$$a_1(N) = a_2(N) = a_3(N) = -(\mu - \lambda)/\nu < 0$$

$\Rightarrow$ Induced chains $\mathcal{L}^{\{1,2\}}$, $\mathcal{L}^{\{2,3\}}$ and $\mathcal{L}^{\{1,3\}}$ are positive recurrent.

Three queues with service interactions: Case 1

1. Consider the case of $\lambda\rho < \delta$, then

$$a_1(\{1,2\}) = a_2(\{2,3\}) = a_3(\{1,3\}) = -\Big(\frac{\delta/\lambda}{1-\rho+\delta/\lambda}\mu - \lambda\Big)/\nu < 0,$$
$$a_2(\{1,2\}) = a_3(\{2,3\}) = a_1(\{1,3\}) = -(\mu-\lambda)/\nu < 0,$$
$$a_3(\{1,2\}) = a_1(\{2,3\}) = a_2(\{1,3\}) = 0.$$

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a_3(\{1,2\}) = a_1(\{2,3\}) = a_2(\{1,3\}) &= 0.
\end{aligned}
$$

$\Rightarrow$ Induced chains $\mathcal{L}^{\{1\}}$, $\mathcal{L}^{\{2\}}$ and $\mathcal{L}^{\{3\}}$ are positive recurrent.

2. It is difficult to get the stationary distributions of $\mathcal{L}^{\{1\}}$, $\mathcal{L}^{\{2\}}$ and $\mathcal{L}^{\{3\}}$, but through some argument we obtain the following.

$$
\begin{aligned}
&a_1(\{1\}) = a_2(\{2\}) = a_3(\{3\}) < -\Big(\frac{\delta/\lambda}{1-\rho+\delta/\lambda}\mu - \lambda\Big)/\nu < 0,\\
&a_2(\{1\}) = a_3(\{1\}) = a_1(\{2\}) = a_3(\{2\}) = a_1(\{3\}) = a_2(\{3\}) = 0.
\end{aligned}
$$

$\Rightarrow$ The three-queue model is positive recurrent.

Three queues with service interactions: Case 2

1. Consider the case of $\lambda\,(\rho - 1/2) < \delta < \lambda\rho$, then

$$a_1(\{1,2\}) = a_2(\{2,3\}) = a_3(\{1,3\}) = -\Big(\frac{\delta/\lambda}{1-\rho+\delta/\lambda}\mu - \lambda\Big)/\nu > 0,$$
$$a_2(\{1,2\}) = a_3(\{2,3\}) = a_1(\{1,3\}) = -(\mu - \lambda)/\nu < 0,$$
$$a_3(\{1,2\}) = a_1(\{2,3\}) = a_2(\{1,3\}) = 0.$$

$\Rightarrow$ Induced chains $\mathcal{L}^{\{1\}}$, $\mathcal{L}^{\{2\}}$ and $\mathcal{L}^{\{3\}}$ are transient.

Three queues with service interactions: Case 2

1. Consider the case of $\lambda\left(\rho-1/2\right)<\delta<\lambda\rho$, then

$$a_1(\{1,2\})=a_2(\{2,3\})=a_3(\{1,3\})=-\Big(\frac{\delta/\lambda}{1-\rho+\delta/\lambda}\mu-\lambda\Big)/\nu>0,$$
$$a_2(\{1,2\})=a_3(\{2,3\})=a_1(\{1,3\})=-(\mu-\lambda)/\nu<0,$$
$$a_3(\{1,2\})=a_1(\{2,3\})=a_2(\{1,3\})=0.$$

$\Rightarrow$ Induced chains $\mathcal{L}^{\{1\}}$, $\mathcal{L}^{\{2\}}$ and $\mathcal{L}^{\{3\}}$ are transient.

2. In this case, it suffices to check the following condition.

$$\left|\frac{a_3(\{1,3\})}{a_1(\{1,3\})}\right|\left|\frac{a_2(\{3,2\})}{a_3(\{3,2\})}\right|\left|\frac{a_1(\{2,1\})}{a_2(\{2,1\})}\right|=\left(\frac{\lambda-\frac{\delta/\lambda}{1-\rho+\delta/\lambda}\,\mu}{\mu-\lambda}\right)^3<1.$$

$\Rightarrow$ The three-queue model is positive recurrent.

3D-RRW-WBP: Positive recurrence

For Case 2, we used the following theorem.

Theorem (3D-RRW-WBP)

Assume the 3D-RRW-WBP satisfies $\boldsymbol{a}(N) < \mathbf{0}$ *and*

$$a_1(\{1,3\}) > 0,\ a_3(\{1,3\}) < 0,$$
$$a_3(\{3,2\}) > 0,\ a_2(\{3,2\}) < 0,$$
$$a_2(\{2,1\}) > 0,\ a_1(\{2,1\}) < 0,$$

then it is positive recurrent if

$$\left|\frac{a_3(\{1,3\})}{a_1(\{1,3\})}\right| \left|\frac{a_2(\{3,2\})}{a_3(\{3,2\})}\right| \left|\frac{a_1(\{2,1\})}{a_2(\{2,1\})}\right| < 1.$$

Summary

1. We considered d-dimensional skip-free RRWs with a background process and obtained sufficient conditions on which they were positive recurrent or transient.
2. Those conditions were represented as the problems to obtain positive definite matrices or vectors satisfying certain conditions.
3. In two-dimensional case, we obtained explicit expressions for the conditions.

Summary

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References

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- Fayolle, G., Malyshev, V.A., and Menshikov, M.V., *Topics in the Constructive Theory of Countable Markov Chains*, Cambridge University Press, Cambridge (1995).

$\Rightarrow$ *RRWs without a background process were studied in these references.*

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3. In two-dimensional case, we obtained explicit expressions for the conditions.

References

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- Fayolle, G., Malyshev, V.A., and Menshikov, M.V., *Topics in the Constructive Theory of Countable Markov Chains*, Cambridge University Press, Cambridge (1995).

$\Rightarrow$ *RRWs without a background process were studied in these references.*

Thank you for your attention.