A SURVEY ON ACCIDENTAL SURFACES IN KNOT COMPLEMENTS

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1. What is an accidental surface?

Let K be a knot in the 3-sphere S^3, and F a properly embedded surface in the exterior $E(K)$ of K in S^3. A non-trivial loop l in F is called an *accidental peripheral* if it is freely homotopic into $\partial E(K)$ in $E(K)$ but not in F. Here, an annulus A connecting l and a loop l' in $\partial E(K)$ is called an *accidental annulus* for l. We define an *accidental surface* as such surface F with an accidental peripheral.

We always assume that surfaces are essential, that is, incompressible, ∂-incompressible and not boundary-parallel. For a two-sided surface F in a 3-manifold M, this means that the induced homomorphism $i_\# : \pi_1(F) \to \pi_1(M)$ and the induced map $i_\# : \pi_1(F, \partial F) \to \pi_1(M, \partial M)$ are injective. The existence of an accidental peripheral causes that $i_\#(\pi_1(F))$ contains an element which is conjugate to some element of the peripheral subgroup $\pi_1(\partial M)$. Thus, in the case that M is hyperbolic, $\rho(i_\#(\pi_1(F)))$ contains an accidental parabolic element, where $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ is a faithful discrete representation.

2. Preparations

Lemma 2.1 ([6])**.** *Let F be an accidental essential two-sided surface properly embedded in $E(K)$. Then there exists an embedded accidental annulus for F.*

3. Accidental closed surfaces

In this section, we treat with accidental closed surfaces. Let S be an accidental closed surface in $E(K)$.

Theorem 3.1 ([3])**.** *All accidental annuli A determine the unique slope $A \cap \partial E(K)$. Moreover, if those slopes are non-meridional, then all accidental annuli are mutually isotopic rel.$S \cup \partial E(K)$.*

Thus, we can define the *accidental slope* of S as a loop $A \cap \partial E(K)$ for an accidental annulus A.

Theorem 3.2 ([1])**.** *Any accidental slope is meridional or integral.*

Thus, there is a meridionaly compressing disk for S or K is isotopic onto S, according to the accidental slope is meridional or integral.

A properly embedded surface in $E(K)$ with non-empty boundary is said to be *strongly essential* if F is essential and at least one component of $E(K) - intN(F; E(K))$ is ∂-irreducible.

Theorem 3.3 ([3])**.** *The following are equivalent.*

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1. *There exists an accidental closed surface S with the accidental slope γ.*
2. *There exists a strongly essential surface F with the boundary slope γ.*

Corollary 3.4 (H. Matsuda)**.** *The boundary slope of a strongly essential surface is meridional or integral.*

The next follows Theorem 3.1.

Corollary 3.5 ([5])**.** *For any strongly essential surface F with an integral boundary slope, $|\partial F| \leq 2$.*

A Seifert surface is said to be *totally knotted* if it is strongly essential. A knot is called a *totally knotted knot* if it bounds a totally knotted Seifert surface.

Theorem 3.6 ([3])**.** *The following are equivalent.*

1. *There exists an accidental closed surface with a separating accidental peripheral.*
2. *K is totally knotted.*

Theorem 3.7 ([3])**.** *Mutually disjoint accidental closed surfaces have the same accidental slope.*

But, there exists a knot which has two accidental closed surfaces with accidental slopes 0 and ∞.

Conjecture 3.8 ([3])**.** *All integral accidental slopes of accidental closed surfaces in a knot complement are coincident.*

It is known that for toroidally alternating knots (this class includes all alternating knots and almost alternating knots), 3-braid knots and Montesinos knots, all closed incompressible surfaces in their complements are meridionally compressible. Hence, these knots satisfy Conjecture 3.8.

Theorem 3.9 ([3])**.** *Composite knots and cable knots satisfy Conjecture 3.8.*

1. *Let K be a composite knot. Then any accidental closed surface in $E(K)$ has a meridional slope.*
2. *Let K be a cable knot. Then any accidental closed surface in $E(K)$ has the integral slope same as the cabling annulus for K.*

Theorem 3.10 ([5])**.** *Let S_1 and S_2 be accidental closed surfaces with accidental slopes γ_1 and γ_2. Then*

$$\Delta(\gamma_1, \gamma_2) \leq min\{-\chi(S_1), -\chi(S_2)\}.$$

4. Accidental surfaces with boundary

In this section, we treat with accidental surfaces with non-empty boundary.

Theorem 4.1 ([6])**.** *If $E(K)$ contains accidental essential surface with boundary slope γ, then $E(K)$ contains an accidental incompressible closed surface with accidental slope γ.*

By Theorems 4.1 and 3.2, the following corollay is obtained.

Corollary 4.2 ([6])**.** *The boundary slope of an accidental essential surface is an integer or $1/0$.*

4.1. **Accidental Seifert surfaces.**

Theorem 4.3 ([2])**.** *Any accidental Seifert surface is non-minimal.*

Theorem 4.4 ([6])**.** *Any accidental incompressible Seifert surface is totally knotted.*

The converse to Theorem 4.4 does not hold.

Theorem 4.5 ([6])**.** *For any positive integer n, there exists a genus one knot such that it bounds a non-accidental, totally knotted Seifert surface of genus n.*

Problem 4.6. *Does there exist a knot which bounds an accidental incompressible Seifert surface?*

For satellite knots, Problem 4.6 is true.

Theorem 4.7 ([6])**.** *There exist infinitely many genus one satellite knots each of which bounds an accidental incompressible Seifert surface of arbitrarily high genus.*

For non-fibered hyperbolic knots, Problem 4.6 is true.

Theorem 4.8 ([7])**.** *There exist infinitely many genus two non-fibered hyperbolic knots each of which bounds an accidental incompressible Seifert surface of arbitrarily high genus.*

4.2. **Accidental surfaces with two or more boundary components.**

Theorem 4.9. *For any positive integer n, there exist infinitely many hyperbolic knots whose exteriors contain an accidental essential surface of arbitrarily high genus with n boundary components.*

This theorem can be proved by taking non-orientable Seifert surfaces or by ∂-annulus compressing Seifert surfaces in the proof of Theorems 4.8.

5. Which knot complement contain an accidental closed surface?

A knot K is said to be *small* if $E(K)$ contains no essential closed surface, and *large* if it is not small. It is known that many knots are large and that torus knots, 2-bridge knots and Montesinos knots with length three are small. It should be noted that Kanji Morimoto proved that twisted torus knots with 2-string twistings are small.

Obviously, knots whose complements contain an accidental closed surface are large. However, does the converse hold?

Problem 5.1. *Does a large knot always contain an accidental closed surface?*

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