ASCENDING NUMBER OF KNOTS AND LINKS

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1. Introduction

Throughout this paper we work in the piecewise linear category. We shall study knots and links in the three-dimensional Euclidean space R^3.

For the purpose of defining a numerical invariant of knots and links, we need to prepare following terms. A link diagram is *ordered* if an order is given to its components. A link diagram is *based* if a basepoint (different from the crossing points) is specified on each component. A link diagram is *oriented* if an orientation is specified on each component.

Let L be a link, and let $\tilde{L}$ be a based ordered oriented link diagram of L. The *descending diagram* of $\tilde{L}$ is obtained as follows, and will be denoted by $d(\tilde{L})$. Beginning at the basepoint of the first component of $\tilde{L}$ and proceeding in the direction specified by the orientation, change the crossings as necessary so that each crossing is first encountered as an over-crossing. Continue this procedure with the remaining components in the sequence determined by the ordering, proceeding from the basepoint in the direction determined by the orientation, changing crossings so that ultimately every crossing is first encountered as an over-crossing. The result is the descending diagram $d(\tilde{L})$ obtained from $\tilde{L}$. An example is shown in Figure1.1. Note that $d(\tilde{L})$ is a diagram of a trivial link.

Figure 1.1

Definition Let L be a link and let $\tilde{L}$ be a based ordered oriented link diagram of L. The *ascending number* of $\tilde{L}$ is defined as the number of different crossings between $\tilde{L}$ and $d(\tilde{L})$, and denoted by $a(\tilde{L})$.

The *(minimum) ascending number* of L is defined as the minimum number of $a(\tilde{L})$ over all based ordered oriented link diagram $\tilde{L}$ of L, and denoted by $a(L)$. The definition and the above note imply that $a(L) \geq u(L)$, where $u(L)$ denotes the unknotting number of L. Accordingly L is a trivial link if and only if $a(L) = 0$.

2. Results

Proposition 2.1. *For a nontrivial knot K, we have*

$$a(K) \leq \left[\frac{c(K)-1}{2}\right].$$

For a link L, we have

$$a(L) \leq \left[\frac{c(L)}{2}\right],$$

where $c(L)$ denotes the minimum crossing number of L, and $[x]$ denotes the greatest integer which does not exceed x.

The following theorem is the fundamental inequality between the ascending number and the bridge number.

Theorem 2.2. *For a n-component link L, we have*

$$a(L) \geq b(L) - n,$$

where $b(L)$ is the bridge number of L.

The following corollary asserts the difference between $a(K)$ and $u(K)$.

Corollary 2.3. *For any nonnegative integer n, there is a knot K such that $a(K) - u(K) \geq n$.*

For the connected sum, we have the following proposition.

Proposition 2.4. *For a knot, the ascending number is subadditive with respect to connected sum $\#$, i.e. $a(K_1 \# K_2) \leq a(K_1) + a(K_2)$.*

It is natural to ask whether the ascending number is additive with respect to the connected sum.

Conjecture 2.5. *For any knots K_1, K_2,*

$$a(K_1 \# K_2) = a(K_1) + a(K_2).$$

The following corollary solves the above question partially.

Corollary 2.6. *Let $K_i (i = 1, 2)$ be a knot. If K_i satisfies $a(K_i) = b(K_i) - 1 (i = 1, 2)$, then*

$$a(K_1 \# K_2) = a(K_1) + a(K_2).$$

In consequence of this, we have the following corollary.

Corollary 2.7. *For any nonnegative integer n, there is a knot K such that $a(K) = n$.*

The following theorem characterizes the ascending number one link.

Theorem 2.8. *Let L be a n-component link. Then the ascending number of L is one if and only if L is*

(twist knot)$\circ O^{n-1}$, if L is completely splittable

or

(Hopf link)$\circ O^{n-2}$, otherwise,

where $\circ$ denotes the split union, and O^n denotes a n-component trivial link. In paticular, the ascending number of a knot K is one if and only if K is a twist knot.

The following theorem determines the ascending number for torus knots.

Theorem 2.9. *Let p and q be coprime positive integers, and let $T_{p,q}$ be a (p, q)-torus knot. Then we have*

$$a(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

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