Accidental surfaces in knot complements

Kazuhiro Ichihara
Department of Mathematics, Tokyo Institute of Technology,
Ookayama 2-12-1, Meguro-ku, Tokyo 152-8551, Japan
ichihara@math.titech.ac.jp

Makoto Ozawa
Department of Mathematics, School of Education, Waseda University,
Nishiwaseda 1-6-1, Shinjuku-ku, Tokyo 169-8050, Japan
ozawa@mn.waseda.ac.jp

Received 27 December 1999
Revised 29 March 2000

ABSTRACT

It is well known that for many knot classes in the 3-sphere, every closed incompressible surface in their complements contains an essential loop which is isotopic into the boundary of the knot exterior. In this paper, we investigate closed incompressible surfaces in knot complements with this property. We show that if a closed, incompressible, non-boundary-parallel surface in a knot complement has such loops, then they determine the unique slope on the boundary of the knot exterior. Moreover, if the slope is non-meridional, then such loops are mutually isotopic in the surface. As an application, a necessary and sufficient condition for knots to bound totally knotted Seifert surfaces is given.

1. Introduction

Let $S$ be a closed, essential (that is, incompressible and non-$\partial$-parallel) surface in the exterior of a knot $K$ in the 3-sphere $S^3$. A nontrivial simple closed curve $l$ on $S$ is called an accidental peripheral when there is an annulus $A$ running from $l$ to a loop on the boundary of the exterior of $K$. We call this annulus $A$ an accidental annulus for $l$, and such an essential surface $S$ with an accidental peripheral an accidental surface. A typical example of accidental surfaces is represented by meridionally compressible surfaces, that is, for each of which there is a compression disk in $S^3$ meeting $K$ at just one point. It is known that for alternating knots ([7]), almost alternating knots ([2]), toroidally alternating knots ([1]), 3-braid knots ([6]), and Montesinos knots ([10]), every closed incompressible surface in their complements is meridionally compressible, and hence accidental.

In this paper, we will be mainly concerned with accidentality of essential surfaces and give a characterization of accidental surfaces.

At first, we show the following.
Theorem 1. (Uniqueness of the slope) Let $S$ be an accidental surface in the exterior of $K$. On the boundary of the exterior of $K$, boundary curves of all accidental annuli for all accidental peripherals on $S$ are mutually isotopic. Moreover, in the case that these curves are non-meridional, all accidental peripherals are mutually isotopic on $S$, and all accidental annuli for these are mutually isotopic relative to $S$ and the boundary of the exterior of $K$.

As a result, boundary curves of accidental annuli for accidental peripherals on $S$ determine the unique slope on the boundary of the exterior of $K$. Let us call this slope the accidental slope for $S$.

Remark 1. By [4, Lemma 2.5.3], we know that any accidental slope is meridional or integral.

Next we consider a characterization of the accidental surfaces. Let $F$ be a surface with non-empty boundary properly embedded in the exterior of $K$. The exterior of $F$ means the closure of the space obtained by removing the regular neighborhood of $F$ from the exterior of $K$. We say that $F$ is strongly essential if $F$ is essential and at least one component of the exterior of $F$ is $\partial$-irreducible. The boundary slope of $F$ is defined as the slope represented by the boundary of $F$. The following results indicate that accidental surfaces and strongly essential surfaces are closely related.

Theorem 2. (Synchronism of accidental and strongly essential surfaces) The following are equivalent.

1. There exists an accidental surface $S$ with the accidental slope $\gamma$.

2. There exists a strongly essential surface $F$ with the boundary slope $\gamma$.

Moreover if one of the conditions holds, the surface in the other condition can be taken so that it is disjoint from the prescribed one.

The following corollary is suggested by H. Matsuda. This follows from Remark 1 and Theorem 2 immediately.

Corollary 1. The boundary slope of a strongly essential surface is either meridional or integral. □

The next theorem can be used to check whether the given essential surface is accidental or not, and to determine its accidental slope if it is accidental.

Theorem 3. (Heredity of the slope)
(A) If there exists an accidental surface $S$ with the accidental slope $\gamma$ in the exterior of $K$, then every essential surface with one or two boundary curves disjoint from $S$ is strongly essential and its boundary slope is $\gamma$.

(B) If there exists an essential surface $F$ with the boundary slope $\gamma$ in the exterior of $K$, then every accidental surface disjoint from $F$ has the accidental slope $\gamma$.

(C) Mutually disjoint accidental surfaces have the same accidental slope.

Remark that we do not assume in Theorem 3 (A) that the surface with boundary is orientable.

As an application of Theorem 3, we give a necessary and sufficient condition for knots to be totally knotted. A Seifert surface $F$ for $K$ is called totally knotted if $F$ is strongly essential in the exterior of $K$. A knot $K$ is called totally knotted if there is a totally knotted Seifert surface for $K$. (Note that our definition differs from that of [13] slightly.)

**Theorem 4. (Separating case)** The following are equivalent.

1. There exists an accidental surface with a separating accidental peripheral.
2. $K$ is totally knotted.

**Remark 2.** Compare the result of [11] that there exists an order zero incompressible closed surface if and only if $K$ bounds a non-free incompressible Seifert surface.

There are two motivations to study the accidental surfaces in knot complements. One of them is a concern with Dehn surgery. For an incompressible but not accidental surface, Y.-Q. Wu proved in [16] that there are at most three surgeries which make it compressible. For a meridionally compressible surface, W. Menasco showed in [7] that if it has two non-parallel accidental annuli, then it remains incompressible after any non-trivial Dehn surgery. On the other hand, many knots have meridionally compressible surface which become compressible after all integral slope Dehn surgery. See for example, [15, Theorem 4.8]. For an accidental surface with an integral accidental slope $\gamma$, it becomes compressible in the manifold obtained by $\gamma'$-Dehn surgery if and only if the geometric intersection number of $\gamma$ and $\gamma'$ is less than zero or one [4, Theorem 2.4.3].

The other is a concern with hyperbolic geometry. In the case that $K$ is hyperbolic, it is known that a closed, essential, but not accidental surface in its complement is quasi-Fuchsian, see [14] and [9] for details. However it is still unknown that what kind of knots have such surfaces in their complements. The first explicit
examples of knots whose complements contain closed quasi-Fuchsian surfaces were
given in [3]. These examples were motivated by the conjecture raised in [8] (also see [5])
that a hyperbolic knot complement contains no closed embedded totally
geodesic surfaces.

2. Proofs

Throughout the sequel, let $K$ be a knot in the 3-sphere $S^3$. We will use the
following notations. The number of connected components of a space $X$ is denoted
by $|X|$, the interior of $X$ by $\text{Int}X$, and the boundary of $X$ by $\partial X$. The regular
neighborhood of a subspace $Y$ in $X$ is denoted by $N(Y;X)$, or simply by $N(Y)$,
and the exterior of $Y$ in $X$, namely the closure of $X - N(Y)$, by $E(Y)$.

In this section, we will prove Theorem 1, 2, 3 and 4 in the following sequence.

**Proof of Theorem 1.** Let $A_1$ and $A_2$ be accidental annuli for accidental peripher-
als $l_1$ and $l_2$ on $S$ respectively. By isotopies of $A_1$ and $A_2$, we may assume that
$A_1 \cap A_2$ consists of essential arcs and loops in both $A_1$ and $A_2$, and assume that
$|A_1 \cap A_2|$ is minimal up to isotopy of $A_1$ and $A_2$. If $A_1 \cap A_2$ has no arc, then
the slopes determined by $A_1$ and $A_2$ coincide. Otherwise, there exists a rectangle
shaped region $\delta$ of $A_2 - A_1$. By the minimality of $|A_1 \cap A_2|$, $\delta$ meets $A_1$ in its both
sides. Therefore, there is an accidental annulus $A_0$ for $S$ coming from the disk $\delta$
and one of the two disks that $\partial \delta$ cuts $A_1$ into. This $A_0$ intersects $A_1$ in a single
essential arc. Then the loop $\partial N((A_1 \cup A_0) \cap S;S)$ bounds a disk $\Delta$ in $E(K)$ such
that $\Delta \cap S = \partial \Delta$. Since $S$ is incompressible in $E(K)$, $\Delta$ shows that $S$ is a peripheral
torus, a contradiction.

Hereafter, suppose that the accidental slope for $S$ is non-meridional. Again let
$A_1$ and $A_2$ be accidental annuli for accidental peripherals $l_1$ and $l_2$ on $S$ respectively.
By the above argument, we may assume that $A_1 \cap A_2$ consists of only loops. First,
we consider the case that $|A_1 \cap A_2| = 0$. Take a compressing disk $D$ for $S$ in $S^3$ so that
$|D \cap K|$ is minimal. Then, we obtain an essential planar surface $P = D \cap E(K)$.
By an isotopy of $P$, we may assume that $P \cap (A_1 \cup A_2)$ consists of essential arcs
in all of $A_1$, $A_2$ and $P$, and that $|P \cap (A_1 \cup A_2)|$ is minimal. Then, there exists a rectangle
shaped disk region $\delta$ of $P - (A_1 \cup A_2)$ whose boundary is a union of an essential
arc of $A_1$, an arc $\alpha$ on $S$ connecting $l_1$ and $l_2$, an essential arc of $A_2$, and
an arc $\beta$ in $\partial E(K)$ connecting $A_1 \cap \partial E(K)$ and $A_2 \cap \partial E(K)$. This $\beta$ is an essential
arc in an annulus $A_0$ of $\partial E(K) - (\partial A_1 \cup \partial A_2)$. By pasting $A_1$, $A_2$, two copies of $\delta$
and a disk $A_0 - \beta$, we can construct a disk $\Delta$ in $E(K)$ such that $\Delta \cap S = \partial \Delta$. Since
$S$ is incompressible in $E(K)$, $\partial \Delta$ bounds a disk $\Delta'$ in $S$ which does not contain
$l_1$, $l_2$ and $\alpha$. This implies that $l_1$ is isotopic to $l_2$ in $S$. By irreducibility of a knot
complement, $\Delta \cup \Delta'$ bounds a 3-ball in $E(K)$. Hence $A_1$ and $A_2$ are mutually
isotopic with respect to $S$ and $\partial E(K)$.

Next, suppose that $|A_1 \cap A_2| \geq 1$, and let $A'_2$ be a component of $A_2 - A_1$ which
meets $\partial E(K)$. Pasting a subannulus of $A_1$ with $A'_2$, we obtain an accidental annulus
$A_2'$ disjoint from $A_1$. By the above argument, $A_2'$ is parallel to $A_1$, and this shows that $A_2'$ is parallel to a subannulus of $A_1$ since $S$ is not a peripheral torus. Therefore we can reduce $|A_1 \cap A_2|$ by isotopy, and eventually we have $|A_1 \cap A_2| = 0$. This completes the proof.  

**Proof of Theorem 2.** $1 \Rightarrow 2$. Let $l$ be an accidental peripheral on $S$, and $A$ the corresponding accidental annulus determining the accidental slope $\gamma$. Cutting $S$ along $l$ and pasting two copies of $A$, we construct a properly embedded surface $F$ with non-empty boundary in $E(K)$. Since $S - l$ is incompressible in $E(K) - A$, $F$ is incompressible in $E(K)$. The incompressibility of $S$ in $E(K)$ implies that at least one of the exterior of $F$ is $\partial$-irreducible. If $F$ is $\partial$-parallel in $E(K)$, then $S$ would be a peripheral torus, a contradiction. Thus $F$ is strongly essential in $E(K)$. By the construction, the boundary slope of $F$ is the same as the accidental slope for $S$, and we can take $F$ and $S$ so that they are disjoint.

$2 \Rightarrow 1$. Let $M$ be the closure of a component of $E(K) - F$ which is $\partial$-irreducible. By pushing $\partial M$ into $\text{Int}E(K)$, we obtain a closed surface $S$ in $E(K)$. Since $M$ is $\partial$-irreducible and $F$ is incompressible in $E(K)$, $S$ is incompressible in $E(K)$. If $S$ is a peripheral torus, then $F$ is $\partial$-parallel in $E(K)$, a contradiction. Finally, by the construction, $S$ is accidental with the accidental slope $\gamma$ and disjoint from $F$.

**Proof of Theorem 3.** (A) Let $S$ be an accidental surface with the accidental slope $\gamma$, $A$ the corresponding peripheral annulus, and $F$ an essential surface with non-empty boundary such that $|\partial F| \leq 2$ and $F$ is disjoint from $S$. We remark that $F$ is also $\partial$-incompressible since $E(K)$ is a compact irreducible $3$-manifold with toral boundary. Since $A$ is incompressible in $E(K) - S$ and $F$ is essential in $E(K)$, we may assume that $A \cap F$ consists of essential arcs and loops in both $A$ and $F$. But, since $F$ does not meet $S \cap A$, it follows that $A \cap F = \emptyset$ or $A \cap F$ consists of only loops. Hence, the boundary slope of $F$ is $\gamma$.

Now, suppose that $F$ is not strongly essential in $E(K)$. At first, we consider the case that $A \cap F = \emptyset$. By assumption, the component $M$ of the exterior $E(F)$ containing $A$ is $\partial$-reducible. Let $D$ be a compressing disk for $\partial M$ in $M$. We may assume that $D \cap S = \emptyset$ because $S$ is incompressible in $E(K)$. Note that $\partial D$ intersects $A \cap E(K)$ since $|\partial F| \leq 2$ and $F$ is incompressible in $E(K)$. Since $A$ is incompressible in $E(K) - S$, we may assume that $D \cap A$ consists of arcs whose ends are in $A \cap \partial E(K)$. Then, by cutting and pasting $D$ along an outermost disk in $A$, we have a new compressing disk $D'$ for $\partial M$ in $M$ with $|D' \cap A| < |D \cap A|$. Repeating this operation, we can get a compressing disk $D$ for $\partial M$ in $M$ with $D \cap A = \emptyset$. This gives a compression disk for $F$, a contradiction.

In the case that $A \cap F$ consists of only loops, we consider the subannulus $A'$ of $A - F$ nearest to $\partial E(K)$. Again, since $F$ is not strongly essential, the component $M$ of $E(F)$ containing $A'$ is $\partial$-reducible. By the same argument, we have a compressing disk $D$ for $\partial M$ in $M$ such that $D \cap A'$ consists of essential arcs in $A'$. Then an
outermost disk on \( D \) gives a \( \partial \)-compressing disk for \( A' \) in \( M \). This implies that \( A' \) is parallel to a subannulus in \( F \) with respect to \( \partial E(K) \) since \( F \) is incompressible in \( E(K) \). Thus we can reduce \( |A \cap F| \) and eventually have \( A \cap F = \emptyset \).

(B) Let \( F \) be an essential surface with the boundary slope \( \gamma \), \( S \) an accidental surface disjoint from \( F \), and \( A \) an accidental annulus for \( S \). By the essentiality of \( F \) and the incompressibility of \( A \) in \( E(K) - S \), \( A \) can be isotoped so that \( F \cap A = \emptyset \) or \( F \cap A \) consists of loops. Thus \( S \) has the accidental slope \( \gamma \).

(C) Suppose that there are mutually disjoint accidental surfaces in \( E(K) \) and let \( S \) be the nearest one to \( K \). By Theorem 2, there exists a strongly essential surface \( F \) disjoint from \( S \) such that its boundary slope equals the accidental slope for \( S \). Since all other surfaces are disjoint from \( F \), the accidental slopes for those are all coincide with that of \( S \) by Theorem 3 (B).

Proof of Theorem 4. (1) \( \Rightarrow \) (2). Suppose that there exists an accidental surface \( S \) with a separating accidental peripheral \( l \), and let \( A \) be the corresponding accidental annulus for \( l \). Then we cut \( S \) along \( l \) and paste the resultant surfaces and \( A \). Since \( l \) separates \( S \), each of obtained surfaces is a Seifert surface \( F \) disjoint from \( S \). From the incompressibility of \( S \) in \( E(K) \), it follows that \( F \) is incompressible. Hence, by Theorem 3 (A), \( F \) is strongly essential, and \( K \) is totally knotted.

(2) \( \Rightarrow \) (1). Suppose that \( K \) is totally knotted, and let \( F \) be a totally knotted Seifert surface for \( K \). Then \( \partial N(F) \) gives an accidental surface with a separating accidental peripheral.

Remark 3. By the above proof, it follows that two Seifert surfaces obtained by cutting \( S \) along \( l \) and pasting \( A \) are strongly essential.

3. Examples

We have a few corollaries which can be used to check whether a given closed incompressible surface is accidental or not. The following one is easily obtained from Theorem 3 (A).

Corollary 2. Let \( K \) be a non-trivial knot, \( S \) a closed incompressible surface in \( S^3 - K \) and \( F \) a closed surface in \( S^3 \) such that \( K \subset F \) and \( F \cap S = \emptyset \). If \( F - K \) is incompressible in \( S^3 - K \) but \( F \cap E(K) \) is not strongly essential in \( E(K) \), then \( S \) is not accidental.
such that $F - K$ is incompressible in $H - K$ but $F - \text{Int}N(K)$ is not strongly essential in $H - \text{Int}N(K)$. Hence, by our corollary, $\partial H$ is not accidental.

**Corollary 3.** If $K$ is composite, then every closed incompressible surface in its complement is either meridionally compressible or not accidental.

**Proof.** Let $F$ be a decomposing sphere for $K$, and $S$ be a closed incompressible surface in the complement of $K$. We may assume that $F \cap S$ consists of essential loops in $F - K$. If $F \cap S \neq \emptyset$, then an innermost disk on $F$ gives a meridionally compressing disk for $S$. If $F \cap S = \emptyset$ and $S$ is accidental, then Theorem 3 (B) says the accidental slope of $S$ is meridional. □

Let $K$ be a knot and $F$ a closed surface in $S^3$. Then $K$ is called *alternating with respect to $F$* if it can be isotoped into a neighborhood $F \times [-1, 1]$ of $F$ so that it has an alternating diagram $\pi(K)$ on $F$ with respect to a projection $\pi : F \times [-1, 1] \to F$. A diagram $\pi(K)$ on $F$ is said to be *prime* if for any simple closed curve $\gamma$ in $F$ intersecting $\pi(K)$ in two non-double points, $\gamma$ bounds a disk in $F$ which intersects $\pi(K)$ in an arc.

**Theorem 5.** ([12]) Let $K$ be an alternating knot with respect to $F$. Suppose that $K$ has a prime alternating diagram $\pi(K)$ on $F$. Then, there exist a closed surface $R$ in $F \times [-1, 1]$ and an isotopy of $K$ so that $R \cup K$, $R - K$ is connected, incompressible in $S^3 - K$, and $R$ bounds a handlebody in $F \times [-1, 1]$. □

By combining this and Corollary 2, we have

**Corollary 4.** If $K$ is alternating with respect to a closed incompressible surface $S$ in $S^3 - K$ such that $\pi(K)$ is prime, then $S$ is not accidental. □

This corollary seems to be a kind of generalization of non-triviality of alternating knots.

**4. Conjecture**

For accidentality, Theorem 3 (C) lets us expect the next property of accidental surfaces.

**Conjecture 1.** All integral accidental slopes of accidental surfaces in a knot complement are coincident.

The assumption “integral” cannot be omitted. The main idea to construct the following example was given by K. Miyazaki. There is a genus one totally knotted knot which admits an essential 2-string tangle decomposition (See Figure 1). It can be shown that the complement of this knot contains a closed, incompressible,
meridionally compressible surface by tubing operations on the 2-string tangle decomposing sphere. Thus there are two accidental surfaces with slopes 0 and \( \infty \) respectively.

![Diagram of a knotted parallel 2-string tangle and a sphere](image)

Figure 1: genus one totally knotted knot which has an essential 2-string tangle decomposition

However, for integral slopes, we have a partial solution.

**Proposition 1.** Let \( K \) be a cable knot in \( S^3 \). Then any accidental surface in \( E(K) \) has the accidental slope same as the cabling annulus for \( K \).

**Proof.** Let \( T \) be a torus containing \( K \), and \( S \) an accidental surface in \( E(K) \) with an accidental annulus \( A \). Let \( V \) be a solid torus bounded by \( T \) in \( S^3 \). We assume that \( |S \cap T| \) is minimal up to isotopy of \( S \).

If \( |S \cap T| = 0 \), then by the incompressibility of \( S \), \( S \subset S^3 - V \). Further, we may assume that \( A \cap T \) consists of essential loops in both \( A \) and \( T - K \). By pasting the subannulus of \( A - T \) meeting \( S \) and a subannulus in \( T \), we obtain an accidental annulus \( A' \) such that \( A' \cap (T \cap E(K)) = \emptyset \). Thus \( S \) is also an accidental surface for a companion knot of \( K \). But, this is a contradiction since the accidental slope of \( S \) is neither meridional nor integral. Otherwise, \( K \) and \( S \) cobound a subannulus of \( T \), and it gives an accidental annulus for \( S \). Then, Theorem 1 assures Proposition 1.

**Acknowledgements**

We would like to thank Prof. Katura Miyazaki for telling us the relationship between accidental surfaces and Dehn surgery and for giving the idea to construct the example in the last section. We also express our thanks to Dr. Hiroshi Matsuda for suggesting Corollary 1.

**References**


