Abstract Let $L$ be an $n$-component Brunnian link and $F$ be a genus $g$ closed surface containing $L$. Then, we show that $g > (n + 3)/3$.

1. Introduction

An $n$-component link $L = C_1 \cup \cdots \cup C_n$ ($n \geq 3$) in the 3-sphere $S^3$ is said to be Brunnian if it is non-trivial but $L - C_i$ is trivial for all $i$ ([1]). Kazuaki Kobayashi observed that the Borromean rings are contained in a genus 3 Heegaard surface of $S^3$, and asked whether it is contained in a genus 2 Heegaard surface of $S^3$. In this article, we answer Kobayashi’s question in the following theorem.

**Theorem 1** Let $L$ be an $n$-component Brunnian link and $F$ be a genus $g$ closed surface containing $L$. Then, $g > (n + 3)/3$ holds.

Theorem 1 shows that the Borromean rings can not be contained in a genus 2 closed surface.

It seems from the proof that the estimation in Theorem 1 is very rough. The author would expect the following.

**Conjecture 1** Let $L$ be an $n$-component Brunnian link and $F$ be a genus $g$ closed surface containing $L$. Then, $g \geq n$ holds.

We note that the inequality of Conjecture 1 is best possible since any $n$-component link can be contained in a genus $n$ closed surface, which is constructed from peripheral tori of the link by $n - 1$ tubings.

2. Proof

**Lemma 1** Let $L = C_1 \cup \cdots \cup C_n$ be an $n$-component Brunnian link. Then, for any component $C_i$ of $L$, there exists an essential tangle decomposing sphere $S_i$ for $L$ such that $S_i$ intersects $L$ only in $C_i$.

**Proof.** Without loss of generality, it is sufficient to show this lemma only for $i = 1$. Since $L - C_1$ is a trivial link, there exists a splitting sphere $S$ for $L - C_1$. We assume that $S$ intersects
$C_1$ minimally among all splitting spheres for $L - C_1$. Then, $S - \text{int}N(L)$ is incompressible and $\partial$-incompressible in $S^3 - \text{int}N(L)$, namely, $S$ is an essential tangle decomposing sphere for $L$. □

By Lemma 1, the Borromean rings admits at least three essential tangle decompositions. In fact, it was shown in Theorem 4 of [2] that the Borromean rings admits exactly three essential tangle decompositions.

Proof of Theorem 1. Let $L = C_1 \cup \cdots \cup C_n$ be an $n$-component Brunnian link and $F$ be a genus $g$ closed surface containing $L$. If $g \leq (n + 3)/3$, then there exists a component of $F - L$ which is an open disk, say $D$, an open annulus, say $A$, or an open pair of pants, say $P$.

If an open disk $D$ exists, then without loss of generality, let $\partial(D \cup C_1) = C_1$. Thus $C_1$ is trivial in the complement of $C_2 \cup \cdots \cup C_n$. Then, since $L - C_1$ is trivial by the Brunnian property of $L$, $L$ is also trivial. This contradicts that $L$ is Brunnian.

If an open annulus $A$ exists, then without loss of generality, let $\partial(A \cup C_1 \cup C_2) = C_1 \cup C_2$. Thus $C_1$ is parallel to $C_2$ in the complement of $C_3 \cup \cdots \cup C_n$. Then, since $L - C_1$ is trivial by the Brunnian property, $L$ is also trivial. This contradicts that $L$ is Brunnian.

If an open pair of pants $P$ exists, then without loss of generality, there are two possibilities;

Case 1 $C_2$ bounds a punctured torus $P' = P \cup C_1 \cup C_2$ in $F$.

Case 2 $C_1 \cup C_2 \cup C_3$ bounds a pair of pants $P \cup C_1 \cup C_2 \cup C_3$ in $F$.

In Case 1, we note that $P' - L$ is incompressible in $S^3 - L$, otherwise at least one of $C_1$ and $C_2$ bounds a disk $D$ in the complement of the rest. This contradicts that $L$ is Brunnian. By Lemma 1, there exists an essential tangle decomposing sphere $S$ for $L$ such that $S$ intersects $L$ in only $C_1$. We assume that $S$ intersects $P'$ minimally up to isotopy of $S$ in the pair $(S^3, L)$. Then, $S \cap P'$ consists of essential loops in $P'$ which are disjoint from $C_2$. Let $\alpha$ be an innermost loop of $S \cap P'$ in $S$ and $\delta$ be the corresponding innermost disk in $S$. By compressing $P'$ along $\delta$, we obtain a disk bounded by $C_2$ in $S^3 - L$. This contradicts that $L$ is Brunnian.

In Case 2, a trivial link $L - (C_2 \cup C_3)$ is obtained from a trivial link $L - C_1$ by a band sum along a band $b \subset P$. By 8.11 Corollary of [3], the band $b$ is trivial, i.e. there exists a 2-sphere containing $L - C_1$ and $b$. Hence, $L$ is trivial and contradicts that $L$ is Brunnian. □

References


Present Address:
Department of Natural Sciences, Faculty of Arts and Sciences, Komazawa University,
1-23-1 Komazawa, Setagaya-ku, Tokyo, 154-8525, Japan
email:w3c@komazawa-u.ac.jp