Essential surfaces derived from knot and link diagrams

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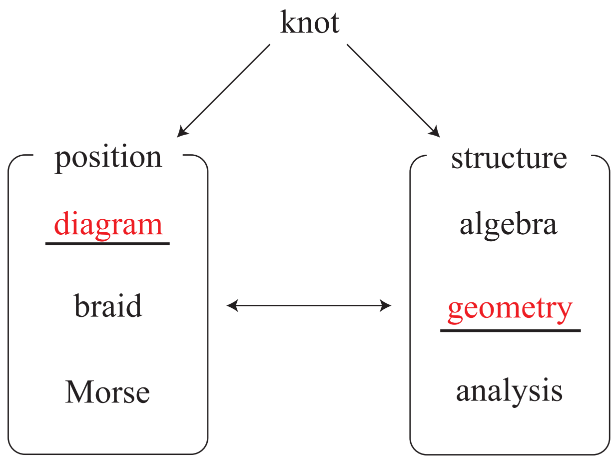
1 How can we recognize knots?



2 Euclid geometry and Knot theory

	Euclid Geometry	Knot theory
object	$(\mathbb{R}^2, \triangle)$	(S^3, K)
group	isometry	homeomorphism
invariant	length, angle	fundamental group
tool	auxiliary line	surface

3 Known knot classes

Let K be a knot in the 3-sphere S^3.

Torus knots

We define an *h-genus* $h(K)$ as the minimal genus of a Heegaard surface of S^3 which contains K. We say that K is a *torus knot* if $h(K) = 1$.

2-bridge knots

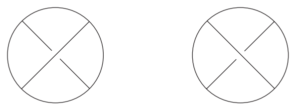
We define a *bridge number* $b(K)$ as the minimal number of maximal points of K with respect to the standard Morse function $h : S^3 \to \mathbb{R}$. We say that K is a *2-bridge knot* if $b(K) = 2$.

Rational tangles and algebraic tangles

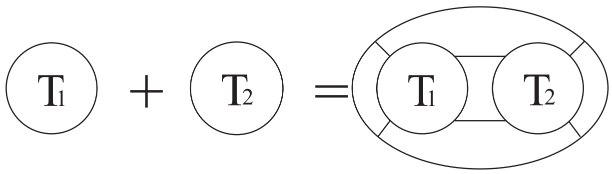
Rational tangles are recursively obtained by tangle sums of rational tangle and $1/1$ or $-1/1$-tangle, and rotations. Every rational tangle has an associated rational number (we call it a *slope*) which is obtained by the formulae $f(T_1 + T_2) = f(T_1) + f(T_2)$ and $f(T^*) = -\dfrac{1}{f(T)}$. Let $R(p/q)$ be a rational tangle of slope p/q.

Algebraic tangles are recursively obtained by tangle sums of two algebraic tangles, and rotations, where rational tangles are regarded as algebraic tangles.

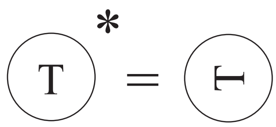
Numerator $N(T)$ of a tangle T is a link obtained by joining the "north" endpoints together and the "south" endpoints also together. *Denominator* $D(T)$ of T is defined similarly by grouping the "east" and "west" endpoints.



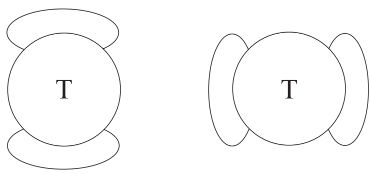
$1/1$-tangle $-1/1$-tangle



tangle sum



rotation



N(T) D(T)

Pretzel knots

We say that K is a *pretzel knot* if $K = N(R(\pm 1/r_1) + R(\pm 1/r_2) + \cdots + R(\pm 1/r_n))$, where each r_i is a natural number.

Montesinos knots

We say that K is a *Montesinos knot* if $K = N(R(p_1/q_1) + R(p_2/q_2) + \cdots + R(p_n/q_n))$, where each p_i is an integer and each q_i is a natural number.

Algebraic knots

We say that K is an *algebraic knot* if $K = N(T)$ for some algebraic tangle T.

Positive knots

Let D be an oriented diagram of K. We say that D is *positive* if all crossing of D has a positive sign, and that K is *positive* if K has a positive diagram.

Alternating knots

Let D be a diagram of K. We say that D is *alternating* if the crossings alternate under, over, under, over, as you travel along D, and that K is *alternating* if K has an alternating diagram.

4 σ-adequate and σ-homogeneous diagrams

Let K be a knot or link in the 3-sphere S^3 and D a connected diagram of K on the 2-sphere S^2 which separates S^3 into two 3-balls, say B_+, B_-. Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be the set of crossings of D. A map $\sigma : \mathcal{C} \to \{+, -\}$ is called a *state* for D. For each crossing $c_i \in \mathcal{C}$, we take a $+$-smoothing or $-$-smoothing according to $\sigma(c_i) = +$ or $-$. See Figure 1. Then, we have a collection of loops $l_1, \ldots, l_m$ on S^2 and call those *state loops*. Let $\mathcal{L}_\sigma = \{l_1, \ldots, l_m\}$ be the set of state loops.

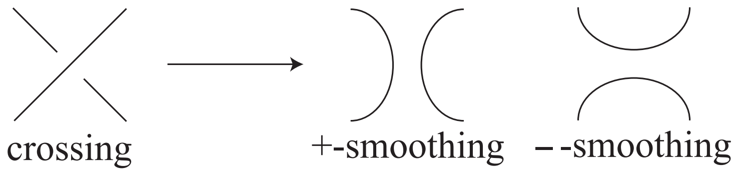


Figure 1: Two smoothings of a crossing

Each state loop l_i bounds a unique disk d_i in B_-, and we may assume that these disks are mutually disjoint. For each crossing c_j and state loops l_i, l_k whose subarcs replaced c_j by $\sigma(c_j)$-smoothing, we attach a half twisted band b_j to d_i, d_k so that it recovers c_j. See Figure 2 for $\sigma(c_j) = +$. In such a way, we obtain a spanning surface which consists of disks $d_1, \ldots, d_m$ and half twisted bands $b_1, \ldots, b_n$ and call this a *σ-state surface*

We construct a graph G_σ with signs on edges from F_σ by regarding a disk d_i as a vertex v_i and a band b_j as an edge e_j which has the same sign $\sigma(c_j)$. We call the graph G_σ a *σ-state graph*. In general, a graph is called a *block* if it is connected and has no cut vertex. It is known that any graph has a unique decomposition into maximal blocks. We say that a diagram D is *σ-adequate* if G_σ has no loop, and that D is *σ-homogeneous* if in each block of G_σ, all edges have a same sign.

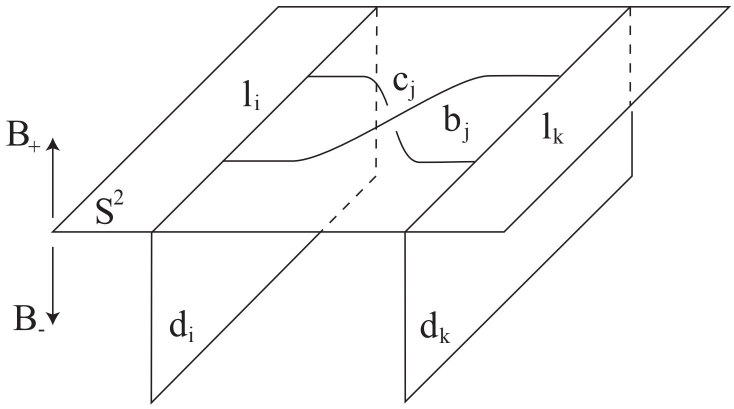


Figure 2: Recovering a crossing by a half twisted band

Positive/negative state and Seifert state

We define the *positive state* σ_+ as a state satisfying $\sigma_+(c_j) = +$ for all j, and the *negative state* σ_- as $\sigma_-(c_j) = -$ for all j.

We define the *Seifert state* $\vec{\sigma}$ as a state determined by an orientation of the knot.

Homogeneous knots and adequate knots

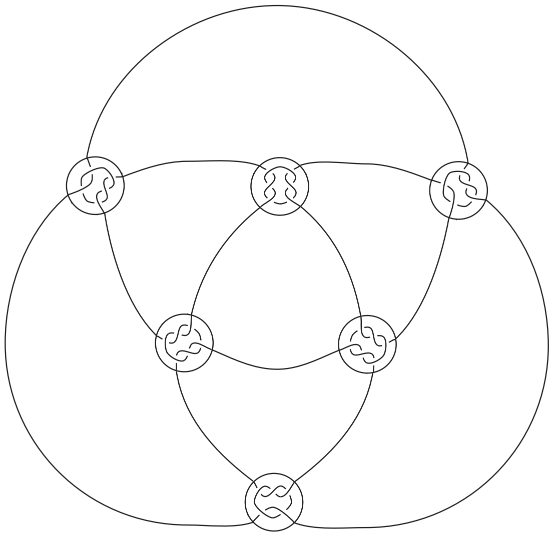
We say that a diagram D is *homogeneous* ([2]) if D is $\vec{\sigma}$-homogeneous for a Seifert state $\vec{\sigma}$. Note that D is automatically $\vec{\sigma}$-adequate since the $\vec{\sigma}$-state surface $F_{\vec{\sigma}}$ is orientable and thus $G_{\vec{\sigma}}$ has no loop.

We say that a diagram D is *semi-adequate* ([5]) if D is σ-adequate for a positive state σ_+ or a negative state σ_-. Note that D is automatically $\sigma_\pm$-homogeneous since $\sigma_\pm(c_j) = \pm$ for all j.

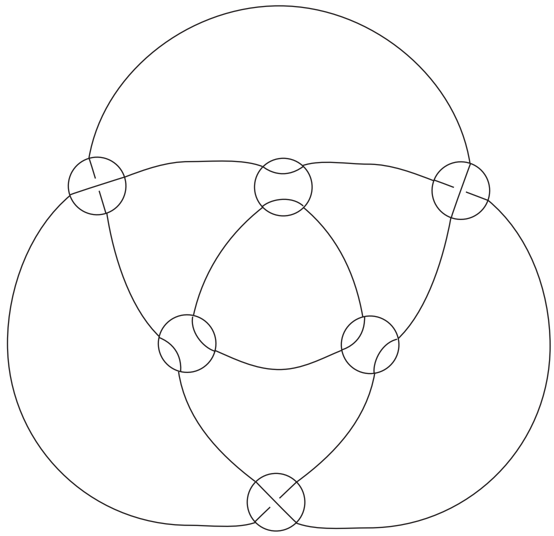
We say that a diagram D is *adequate* ([14]) if D is σ-adequate for both of a positive state σ_+ and a negative state σ_-. Note also that D is automatically $\sigma_\pm$-homogeneous since $\sigma_\pm(c_j) = \pm$ for all j.

5 algebraically alternating diagrams

Let $S^2 \subset S^3$ be the standard 2-sphere and G a connected quadrivalent graph in S^2 which has no bigon. We obtain a knot or link diagram $\tilde{K}$ by substituting algebraic tangles for vertices of G. After such an operation, we substitute each algebraic tangle (B, T) for a rational tangle of slope 1, -1, 0 or ∞ if the slope of (B, T) is positive, negative, 0 or ∞ respectively (fixing four points of ∂T). The resultant knot or link diagram is said to be *basic* and denoted by $\tilde{K}_0$. Then we say that $\tilde{K}$ is *algebraically alternating* if $\tilde{K}_0$ is alternating, and K is *algebraically alternating* if K has an algebraically alternating diagram.



$\tilde{K}$: algebraically alternating link diagram



$\tilde{K}_0$: the basic diagram of $\tilde{K}$

6 Hasse diagram of various knot classes

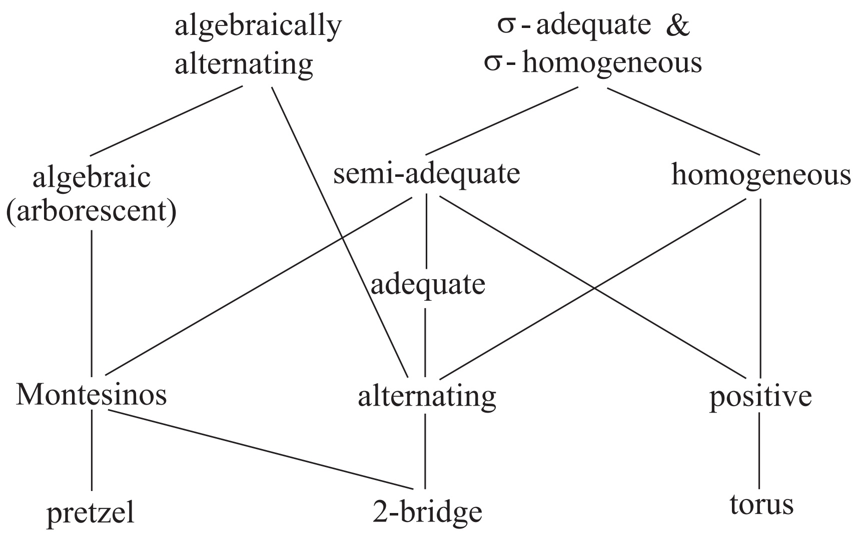


Figure 3: The Hasse diagram for the set of knot diagrams partially ordered by inclusion

7 Essential surfaces derived from knot and link diagrams

We can construct essential surfaces and restrict on position of surfaces from knot diagrams.

7.1 Construction of surfaces

Theorem 7.1 ([1], [11]). *If a diagram D is alternating, then both checkerboard surfaces are essential.*

Theorem 7.2 ([3], [7], [4]). *If a diagram D is alternating, then the canonical Seifert surface is essential (minimal genus).*

Theorem 7.3 ([2]). *If a diagram D is homogeneous, then the canonical Seifert surface is essential (minimal genus).*

Theorem 7.4 ([12]). *If a diagram D is σ-adequate and σ-homogeneous for a state σ, then the σ-state surface is essential.*

7.2 Restriction on position of surfaces

We define the *waist* $w(F)$ of a closed incompressible surface $F \subset S^3 - K$ as follows.

$$w(F) = \min\{\#(D \cap K) | D \text{ is a compressing disk for } F \text{ in } S^3\}$$

We note that $w(F) \leq 2b(K)/3$ (this needs some arguments).

Theorem 7.5 ([8]). *Let K be a Montesinos knot and $F \subset S^3 - K$ a closed incompressible surface. Then $w(F) = 1$.*

Theorem 7.6 ([6]). *Let K be an alternating knot and $F \subset S^3 - K$ a closed incompressible surface. Then $w(F) = 1$.*

Theorem 7.7 ([13]). *Let K be an algebraically alternating knot and $F \subset S^3 - K$ a closed incompressible surface. Then $w(F) = 1$.*

For a closed surface F in $S^3 - K$, let $i : F \to S^3 - K$ be the inclusion map and let $i_* : H_1(F) \to H_1(S^3 - K)$ be the induced homomorphism. Since $Im(i_*)$ is a subgroup of a cyclic group $H_1(S^3 - K)$ generated by a meridian, there is an integer m such that $Im(i_*) = m\mathbb{Z}$. Then we define $o(F) = m$.

We note that if $w(F) = 1$, then $o(F) = \pm 1$.

Theorem 7.8 ([10]). *Let K be a positive knot and $F \subset S^3 - K$ a closed incompressible surface. Then $o(F) \neq 0$*

Theorem 7.9 ([9]). *The following conditions are equivalent.*

1. *There exists a non-free incompressible Seifert surface S for K.*
2. *There exists a closed incompressible surface $F \subset S^3 - K$ with $o(F) = 0$.*

Moreover, if these conditions hold, then we can take S and F so that they are disjoint.

Corollary 7.10. *Let K be an algebraically alternating knot or a positive knot. Then K cannot bound a non-free incompressible Seifert surface.*

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