Closed incompressible surfaces of genus two in 3-bridge knot complements

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**Bridge number** $b(K)$ **and genus** $g(F)$

$K \subset S^3$ : knot

$F \subset S^3 - K$ : closed incompressible surface

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<th>$g(F)$</th>
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- Hatcher-Thurston
- This work
- Schubert
Meridionally compression and tubing

$F$ is incompressible $\Rightarrow F'$ is also incompressible

$F'$ is incompressible $\Rightarrow F$ may not be incompressible
The case that $g(F') = 2$
Theorem I

$K$: 3-bridge knot or link

$F \subset S^3 - K$: closed incompressible and meridionally incompressible surface of genus two

$\Rightarrow$

I-a  I-b  I-c
Theorem II

$K$: 3-bridge knot or link

$F \subset S^3 - K$: closed incompressible and meridionally incompressible surface of genus one

$\Rightarrow$

\[\text{II-a} \quad \text{II-b} \quad \text{II-c}\]
Theorem III

\( K : 3 \)-bridge knot or link

\( F \subset S^3 - K \): closed incompressible and meridionally incompressible surface of genus zero

\( \Rightarrow \)

III-a

III-b
Corollary

Any essential 2-string tangle decomposing sphere for 3-bridge knots bounds a length 2 or 3 Montesinos tangle.
The case that $g(F) \geq 3$

Problem [Hayashi, 1996]
Does there exist closed incompressible and meridionally incompressible surface of genus greater than 2 in the complement of 3-bridge knots?

Theorem IV
Yes, for all $g \geq 2$.

Theorem [Muñoz-Coto, 2004]
There exists a hyperbolic 3-bridge knot which contains quasi-Fuchsian surfaces of arbitrarily high genus.
totally knotted spatial graph
Proof of Theorem I, II and III

Let $f : S^3 \rightarrow \mathbb{R}$ be a Morse function with two critical points.

Put $K$ in a bridge position with respect to $f$.

Let $F$ be a closed incompressible and meridionally incompressible surface.

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Lemma A

One of the following holds.

1. $K$ is a split link.
2. $K$ is not thin position.
3. After an isotopy of $K$ and $F$, there exists a level sphere $S = f^{-1}(x)$ such that each component of $S \cap F$ is essential in both $S - K$ and $F - K$. 
Lemma B

Let \((B,T)\) be a trivial \(n\)-string tangle and \(P\) an incompressible surface in \((B,T)\). Then, one of the following holds.

1. \(P\) is a disk with \(P \cap T = \emptyset\) and separates \((B,T)\) into two trivial tangles.
2. \(P\) is a disk with \(|P \cap T| = 1\) and separates \((B,T)\) into two trivial tangles.
3. \(P\) is \(\partial\)-compressible.

Using Lemma B inductively, we can classify incompressible and meridionally incompressible surfaces in the trivial 3-string tangle as follows.
By Lemma A and the classification of incompressible and meridionally incompressible surfaces in the trivial 3-string tangle, we obtain Theorem I, II and III.

I-a