Closed incompressible surfaces of genus two in 3-bridge knot complements

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Bridge number $b(K)$ and genus $g(F)$

$K \subset S^3$: knot

$F \subset S^3 - K$: closed incompressible surface

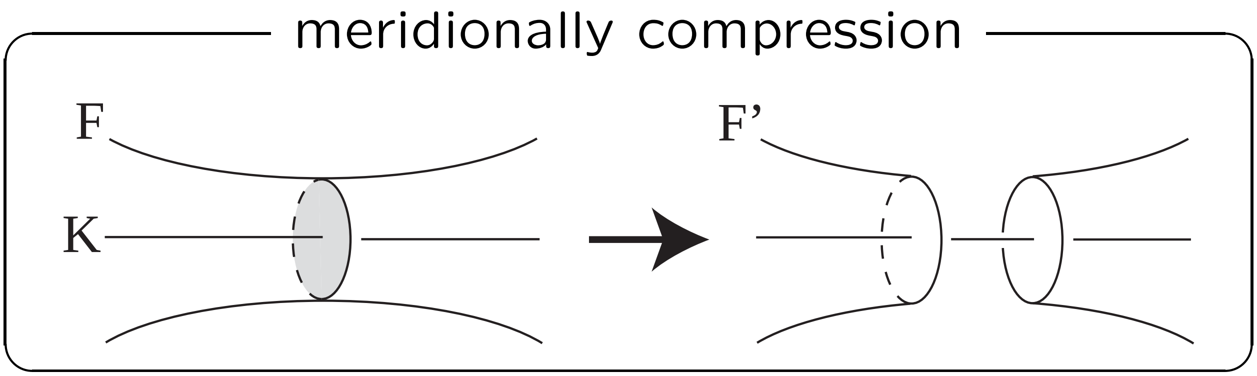
b(K) \ g(F)	1	2	3	4	5	$\cdots$
1	ϕ	ϕ	ϕ	ϕ	ϕ	
2	ϕ	ϕ	ϕ	ϕ	ϕ	
3						
4						
5						
$\vdots$						

← Hatcher -Thurston (row 2)

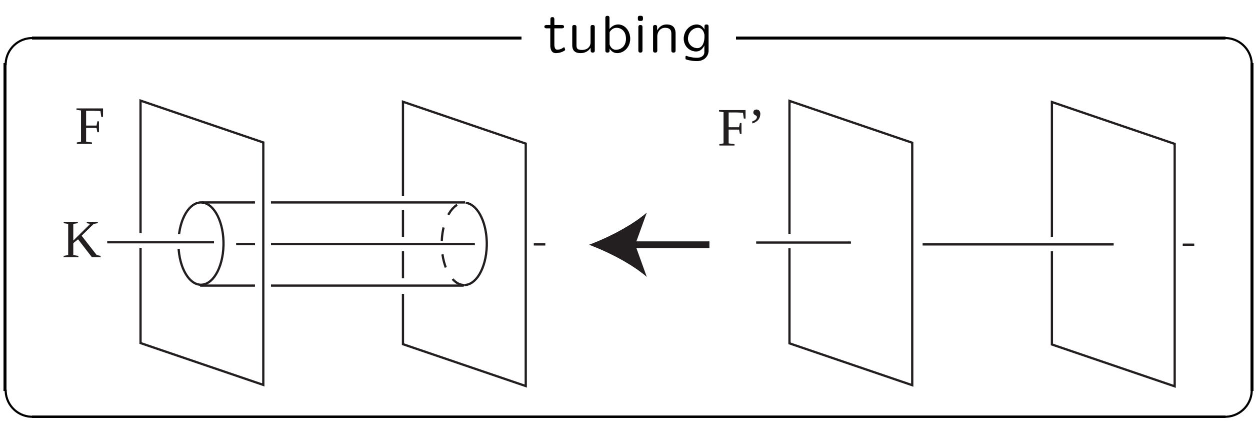
← This work (row 3)

↑ Schubert (column 1)

Meridionally compression and tubing

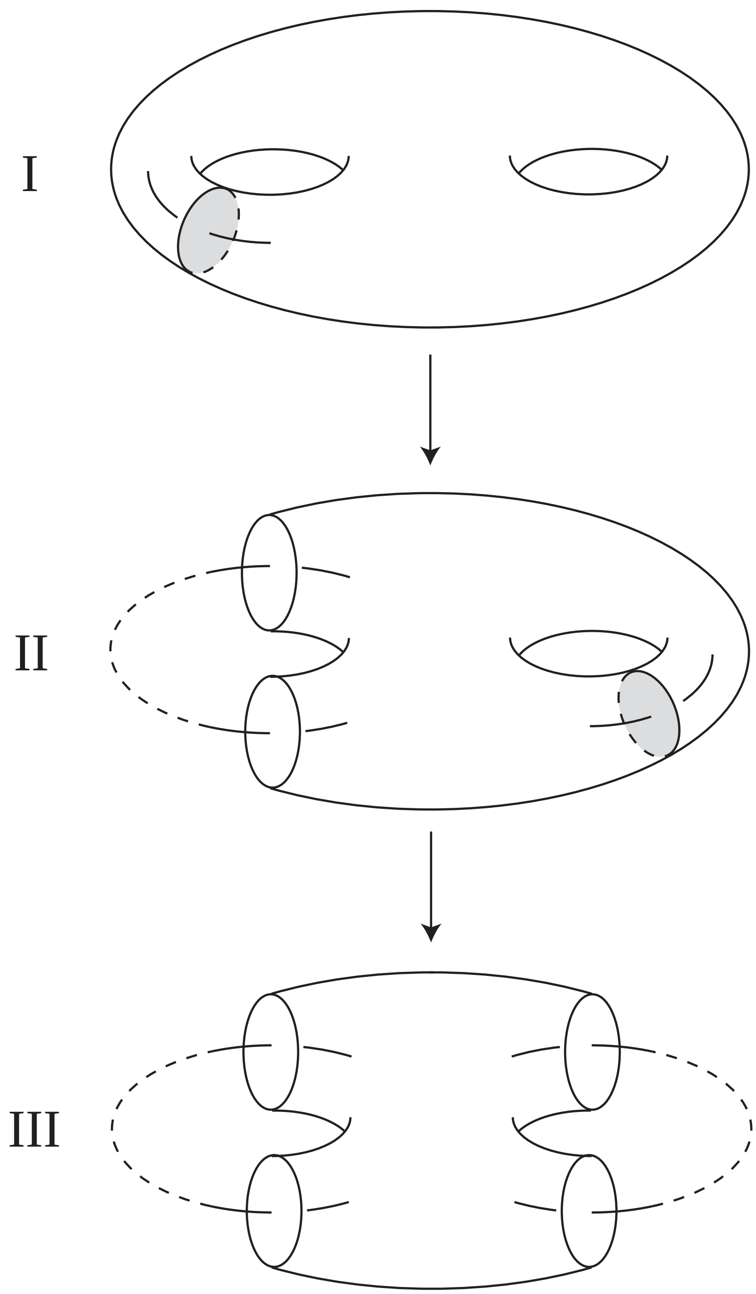


F is incompressible $\Rightarrow$ F' is also incompressible



F' is incompressible $\Rightarrow$ F may not be incompressible

The case that $g(F) = 2$

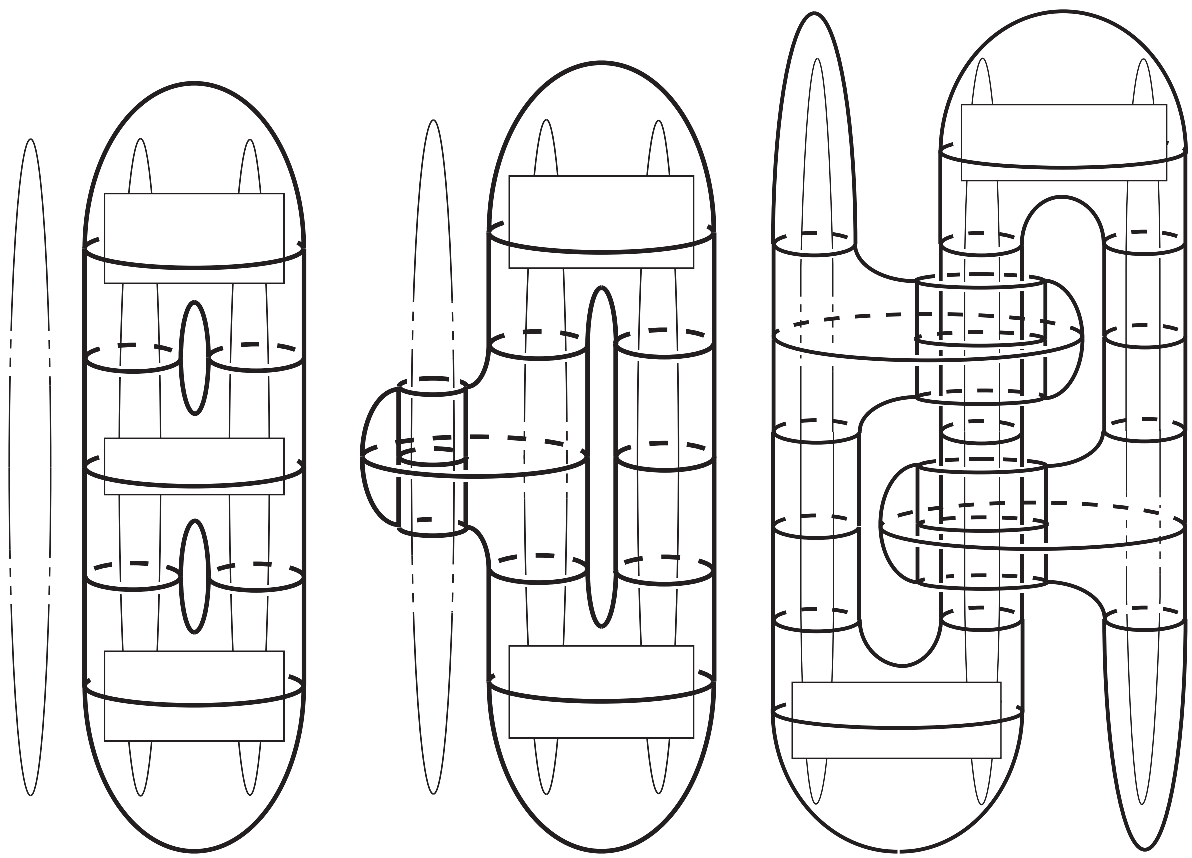


Theorem I

K : 3-bridge knot or link

$F \subset S^3 - K$: closed incompressible and meridionally incompressible surface of genus two

$\Rightarrow$



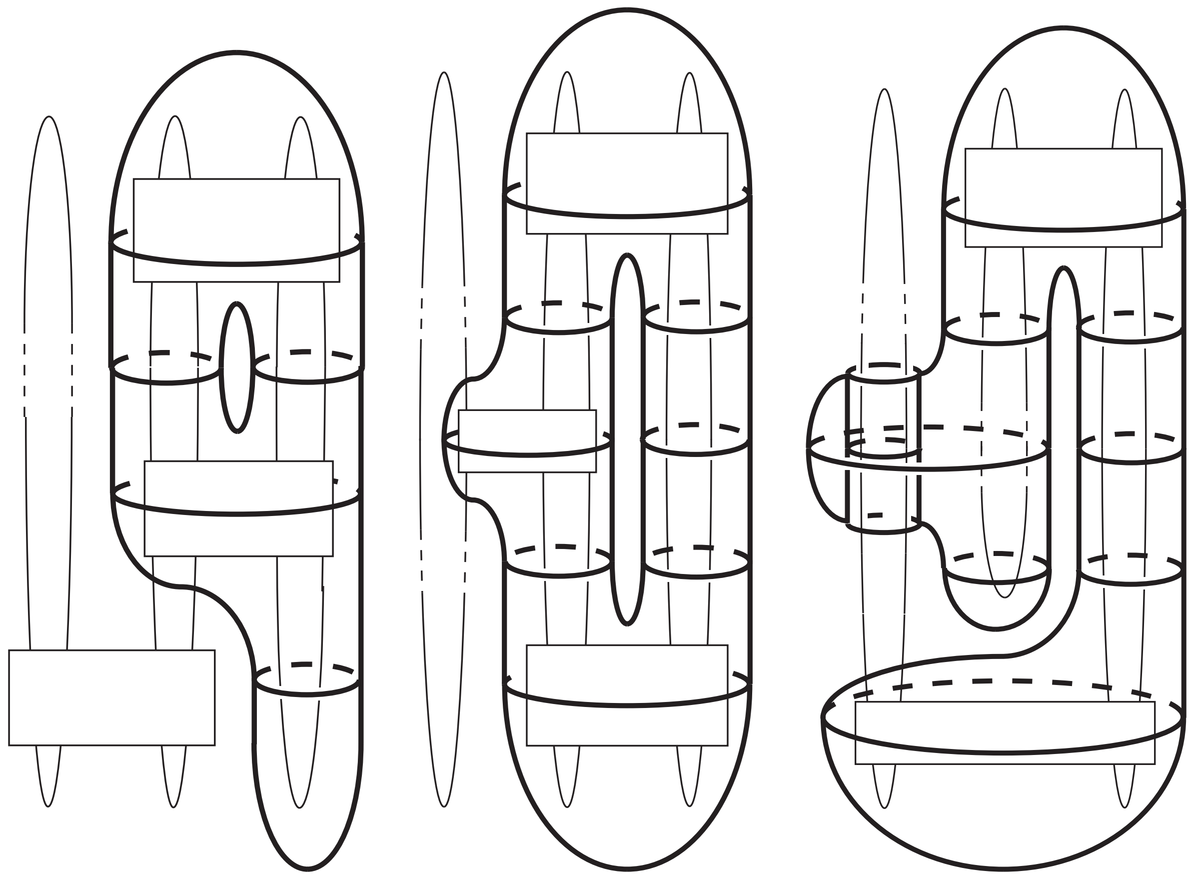
I-a I-b I-c

Theorem II

K : 3-bridge knot or link

$F \subset S^3 - K$: closed incompressible and meridionally incompressible surface of genus one

$\Rightarrow$



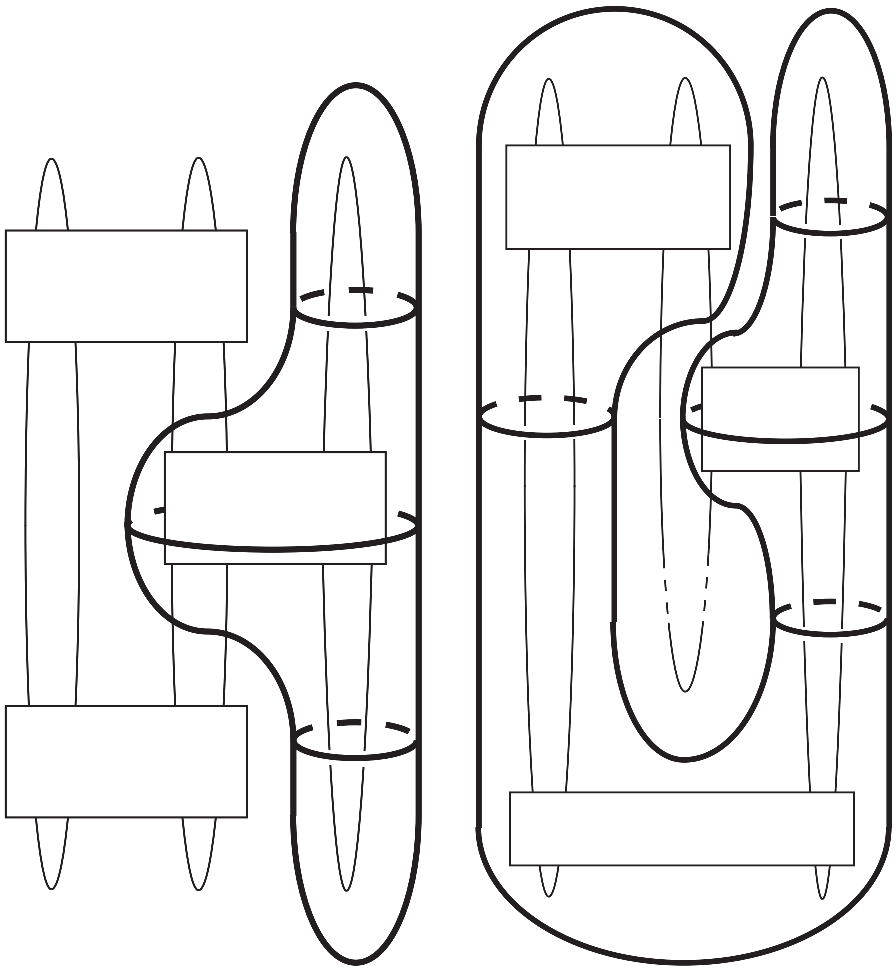
II-a II-b II-c

Theorem III

K : 3-bridge knot or link

$F \subset S^3 - K$: closed incompressible and meridionally incompressible surface of genus zero

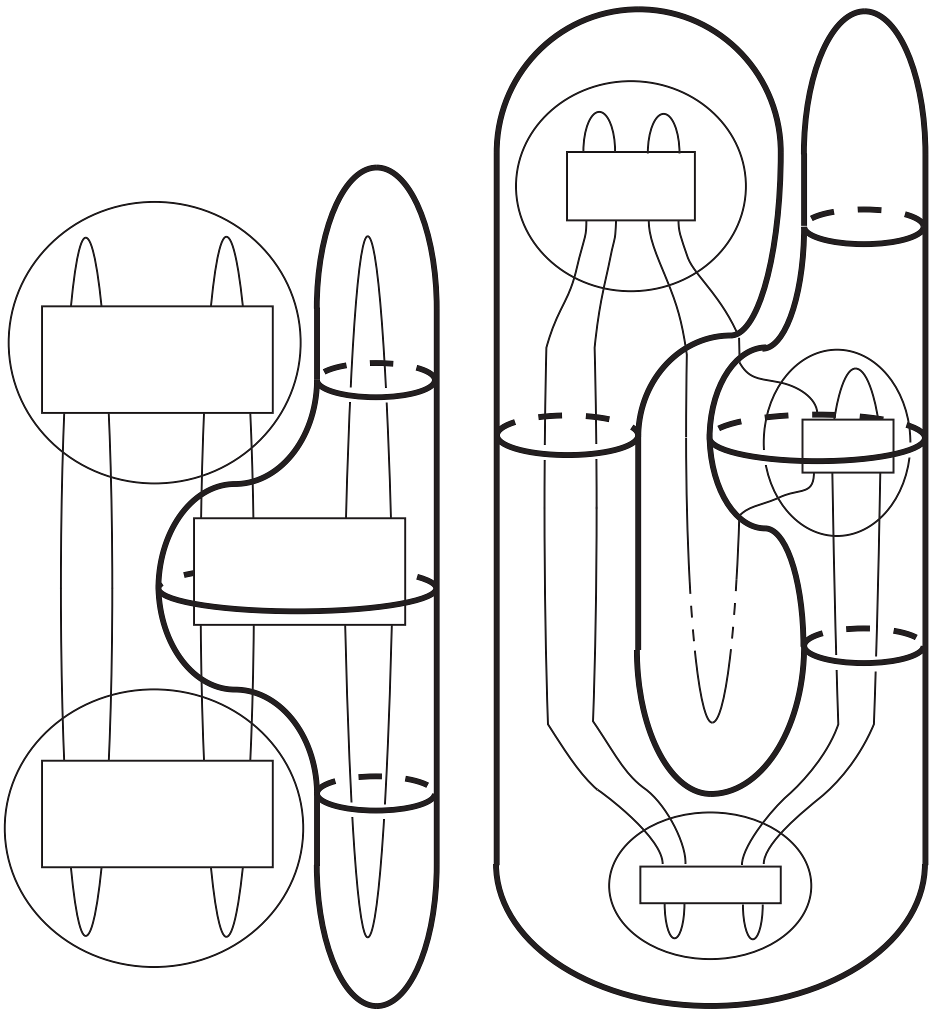
$\Rightarrow$



III-a III-b

Corollary

Any essential 2-string tangle decomposing sphere for 3-bridge knots bounds a length 2 or 3 Montesinos tangle.



III-a III-b

The case that $g(F) \geq 3$

Problem [Hayashi, 1996]

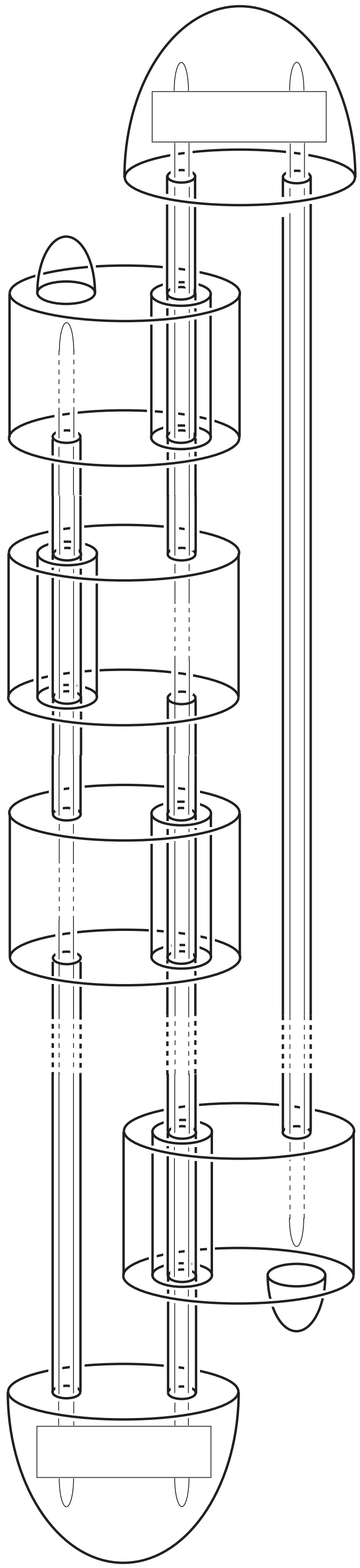
Does there exist closed incompressible and meridionally incompressible surface of genus greater than 2 in the complement of 3-bridge knots?

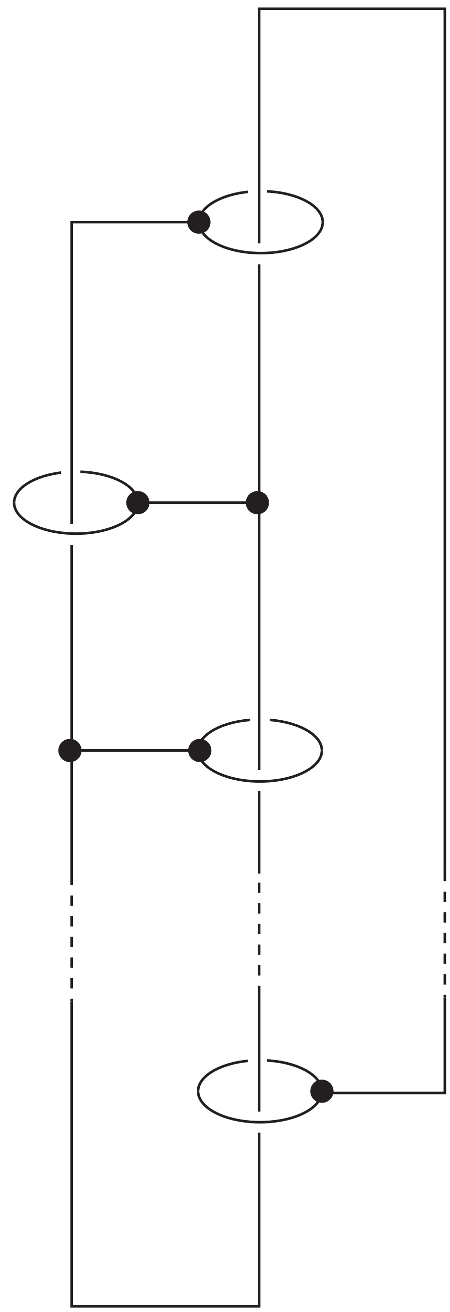
Theorem IV

Yes, for all $g \geq 2$.

Theorem [Muñoz-Coto, 2004]

There exists a hyperbolic 3-bridge knot which contains quasi-Fuchsian surfaces of arbitrarily high genus.





totally knotted
spatial graph

Proof of Theorem I, II and III

Let $f : S^3 \to \mathbb{R}$ be a Morse function with two critical points.

Put K in a bridge position with respect to f.

Let F be a closed incompressible and meridionally incompressible surface.

Lemma A

One of the following holds.

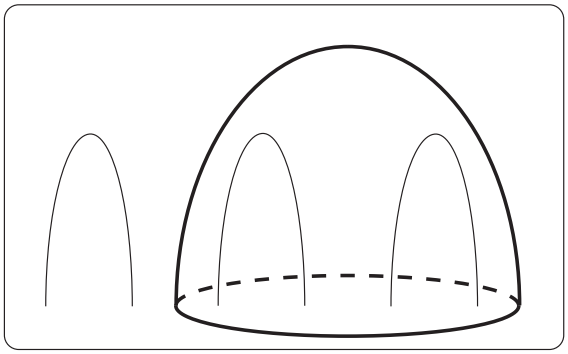
1. K is a split link.
2. K is not thin position.
3. After an isotopy of K and F, there exists a level sphere $S = f^{-1}(x)$ such that each component of $S \cap F$ is essential in both $S - K$ and $F - K$.

Lemma B

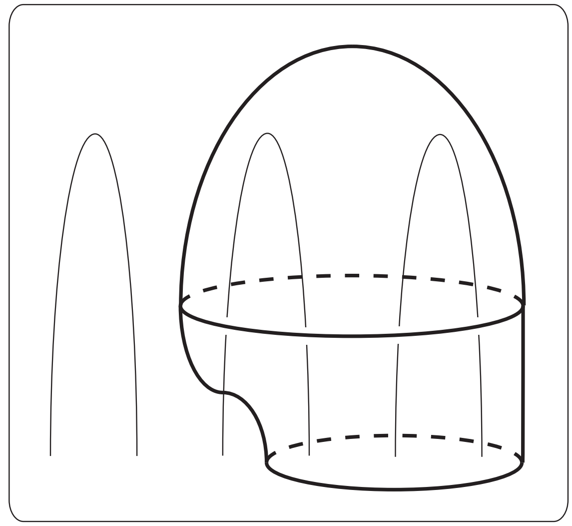
Let (B,T) be a trivial n-string tangle and P an incompressible surface in (B,T). Then, one of the following holds.

1. P is a disk with $P \cap T = \emptyset$ and separates (B,T) into two trivial tangles.
2. P is a disk with $|P \cap T| = 1$ and separates (B,T) into two trivial tangles.
3. P is ∂-compressible.

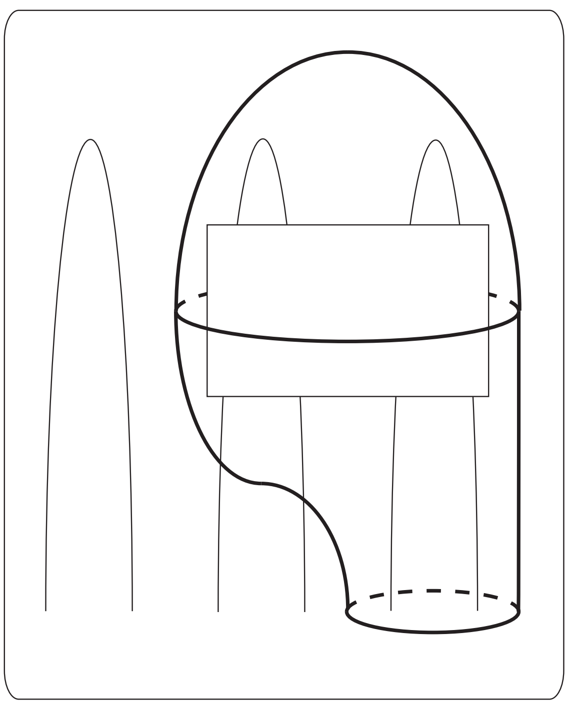
Using Lemma B inductively, we can classify incompressible and meridionally incompressible surfaces in the trivial 3-string tangle as follows.



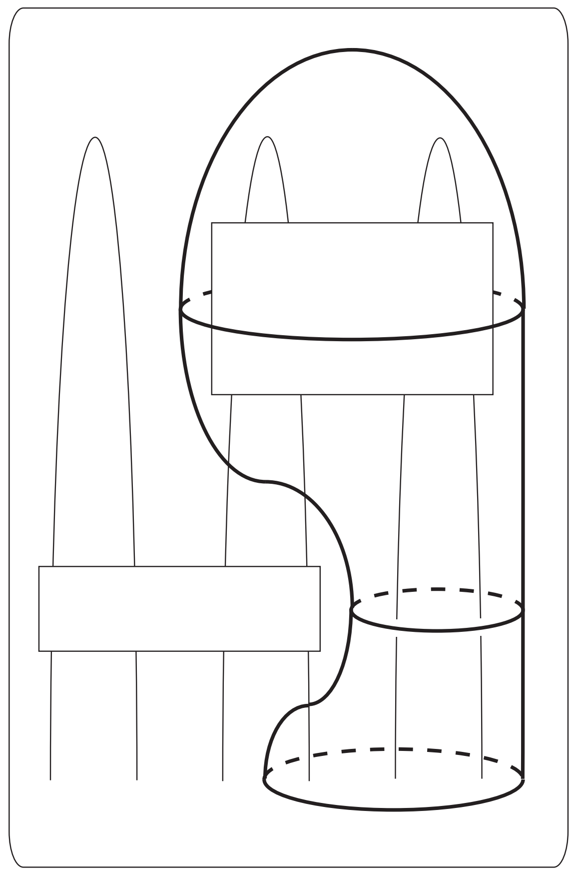
disk(0)



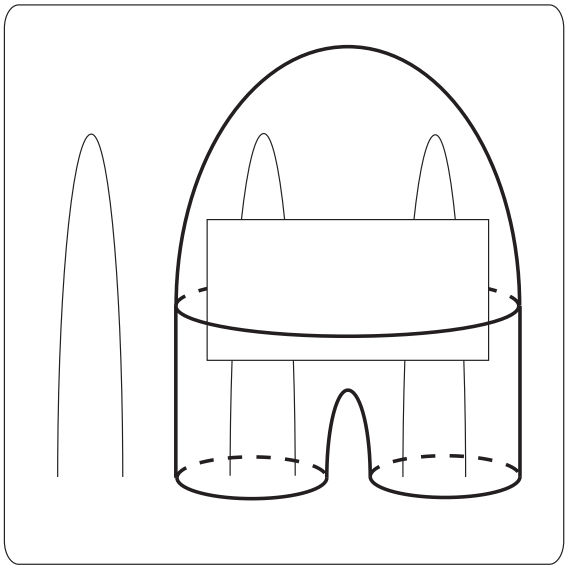
disk(1)



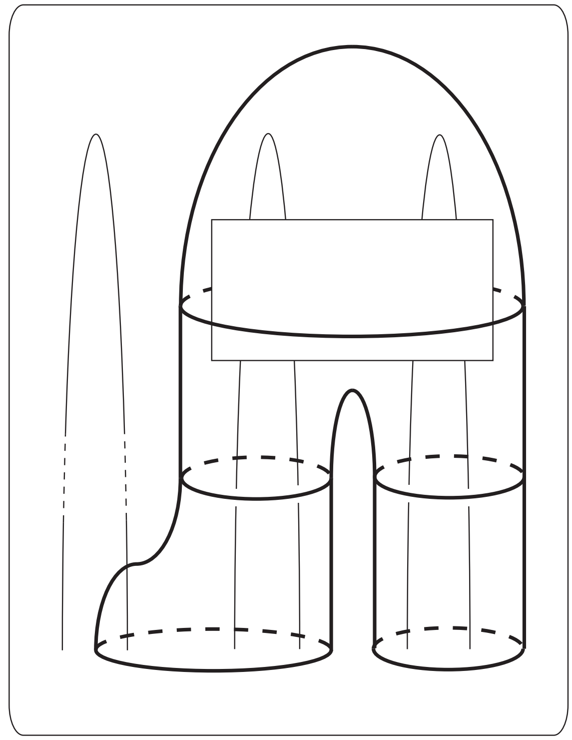
disk(2)



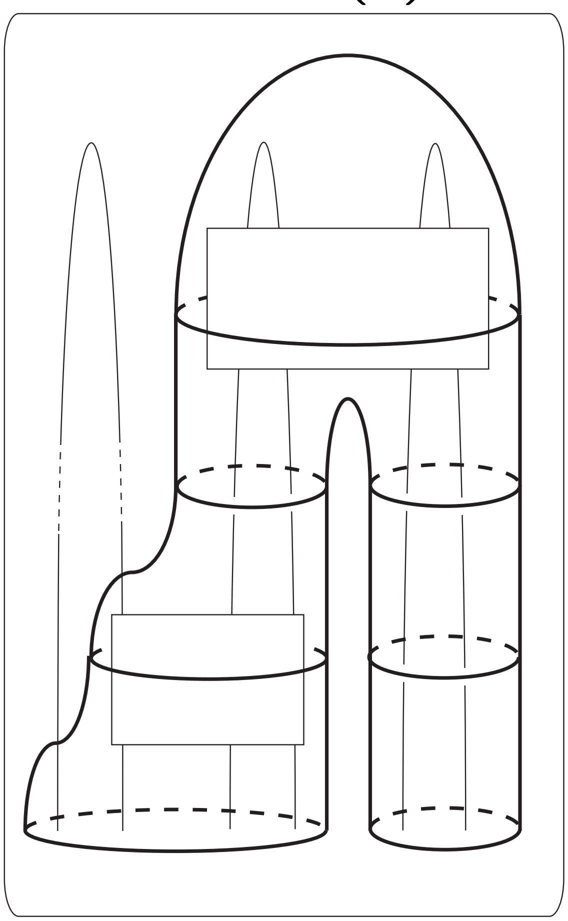
disk(3)



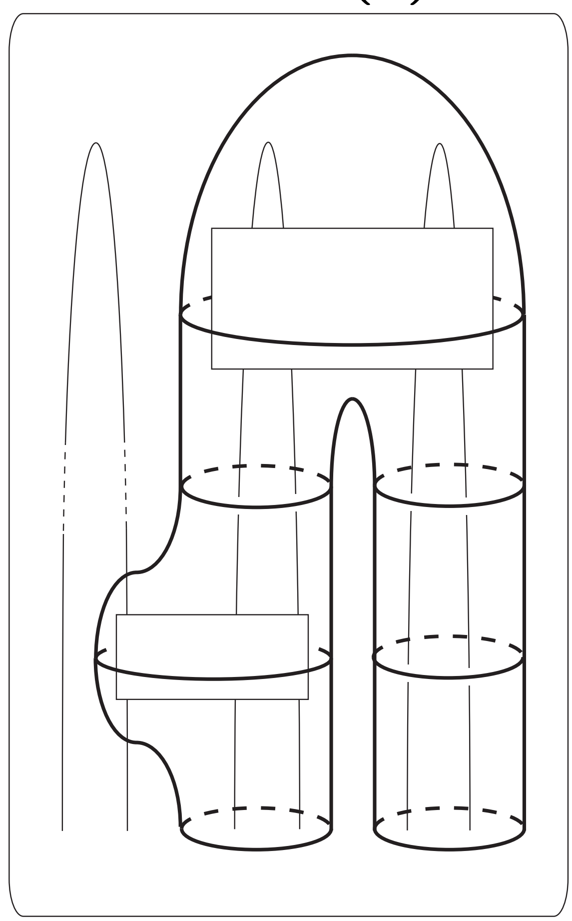
annulus(0)



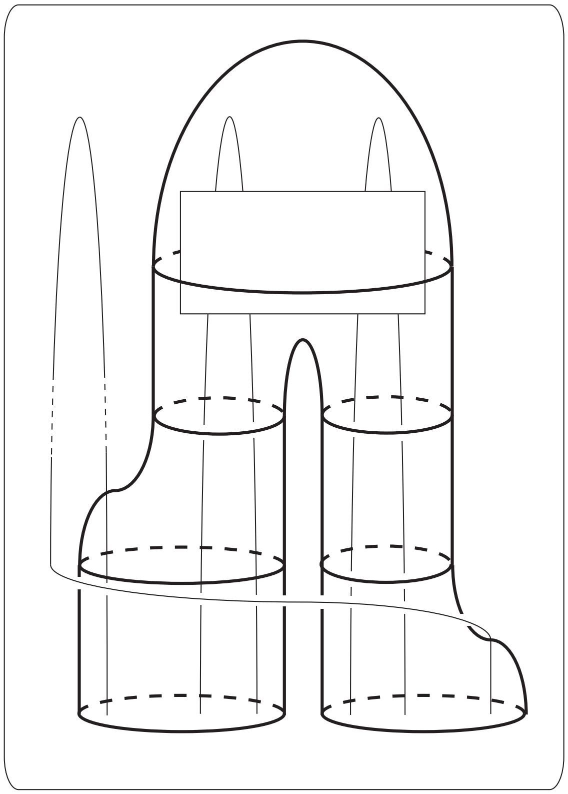
annulus(1)



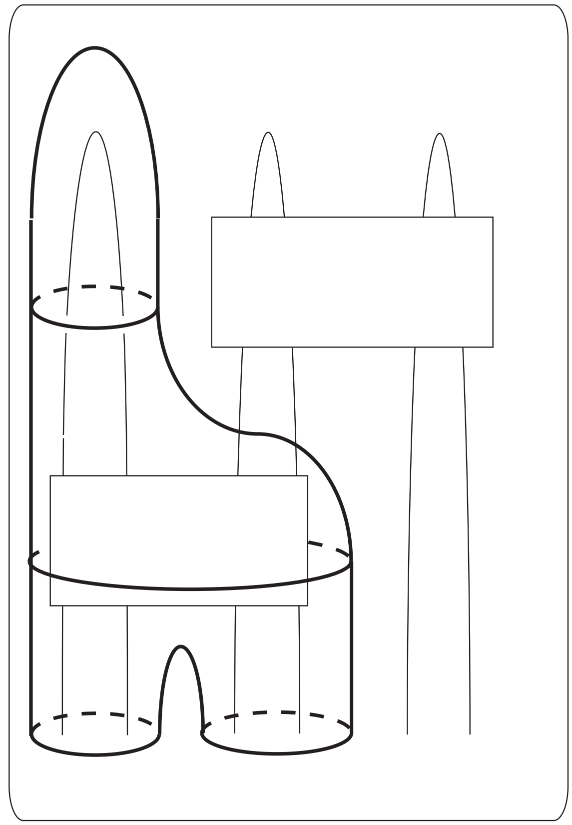
annulus(2)-a



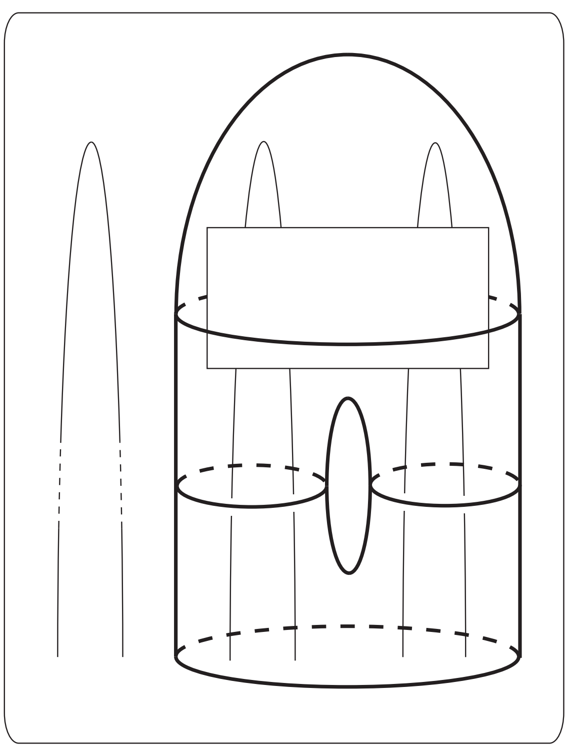
annulus(2)-b



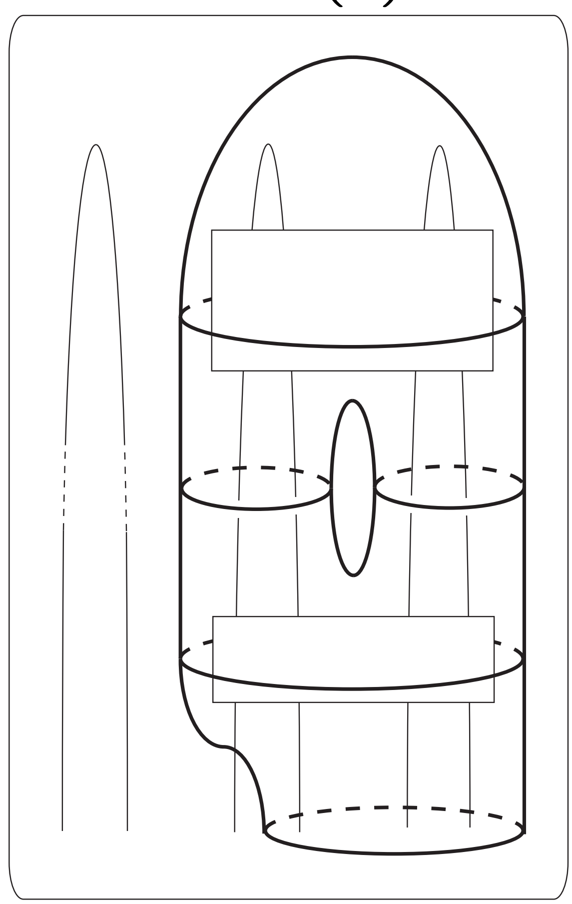
annulus(2)-c



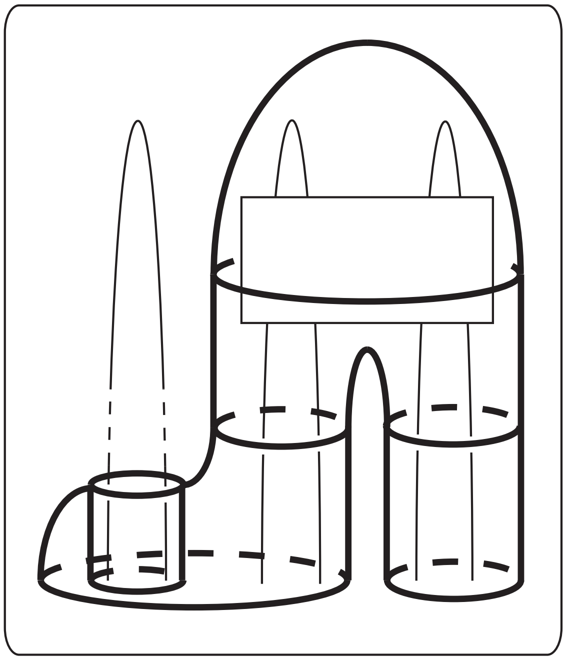
annulus(2)-d



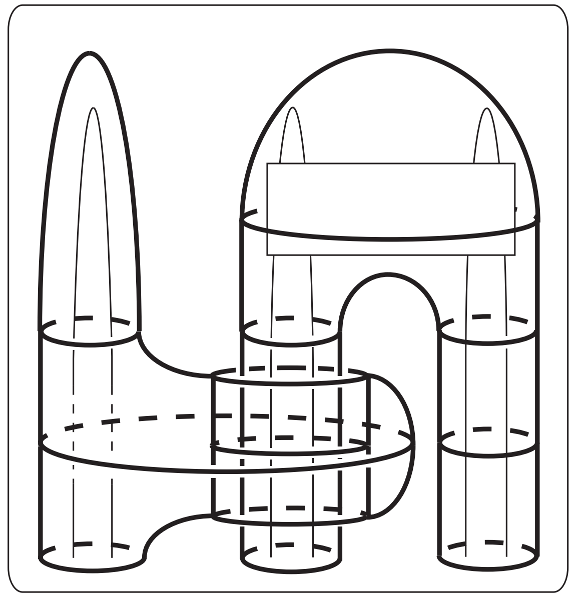
torus(0)



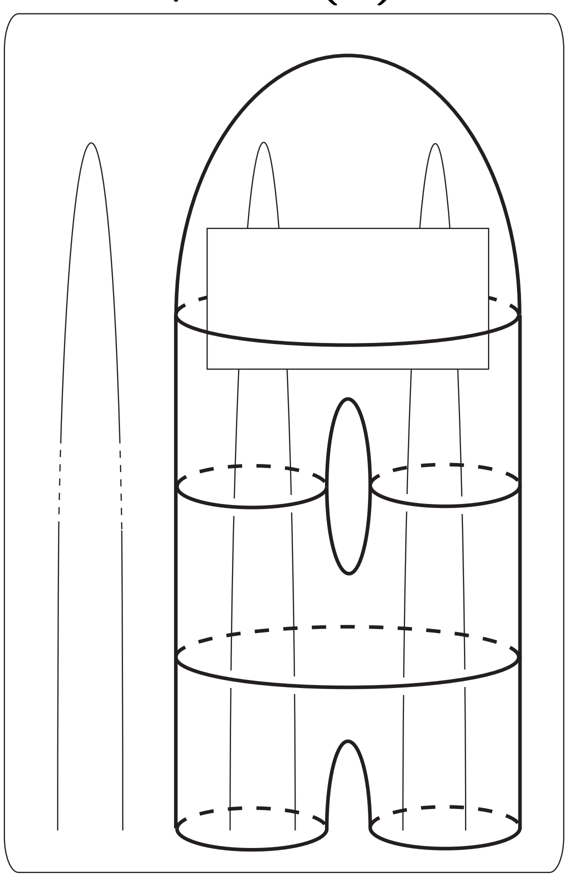
torus(1)



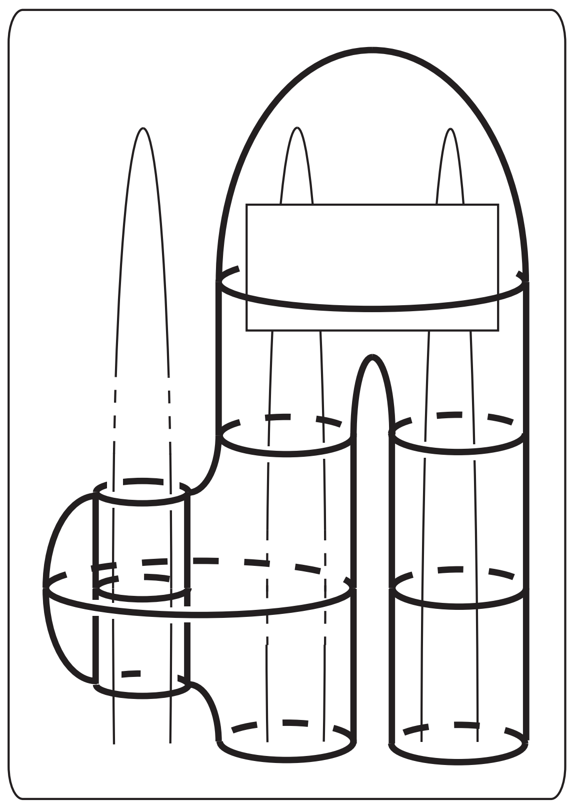
pants(0)



annulus(0)∪torus(0)



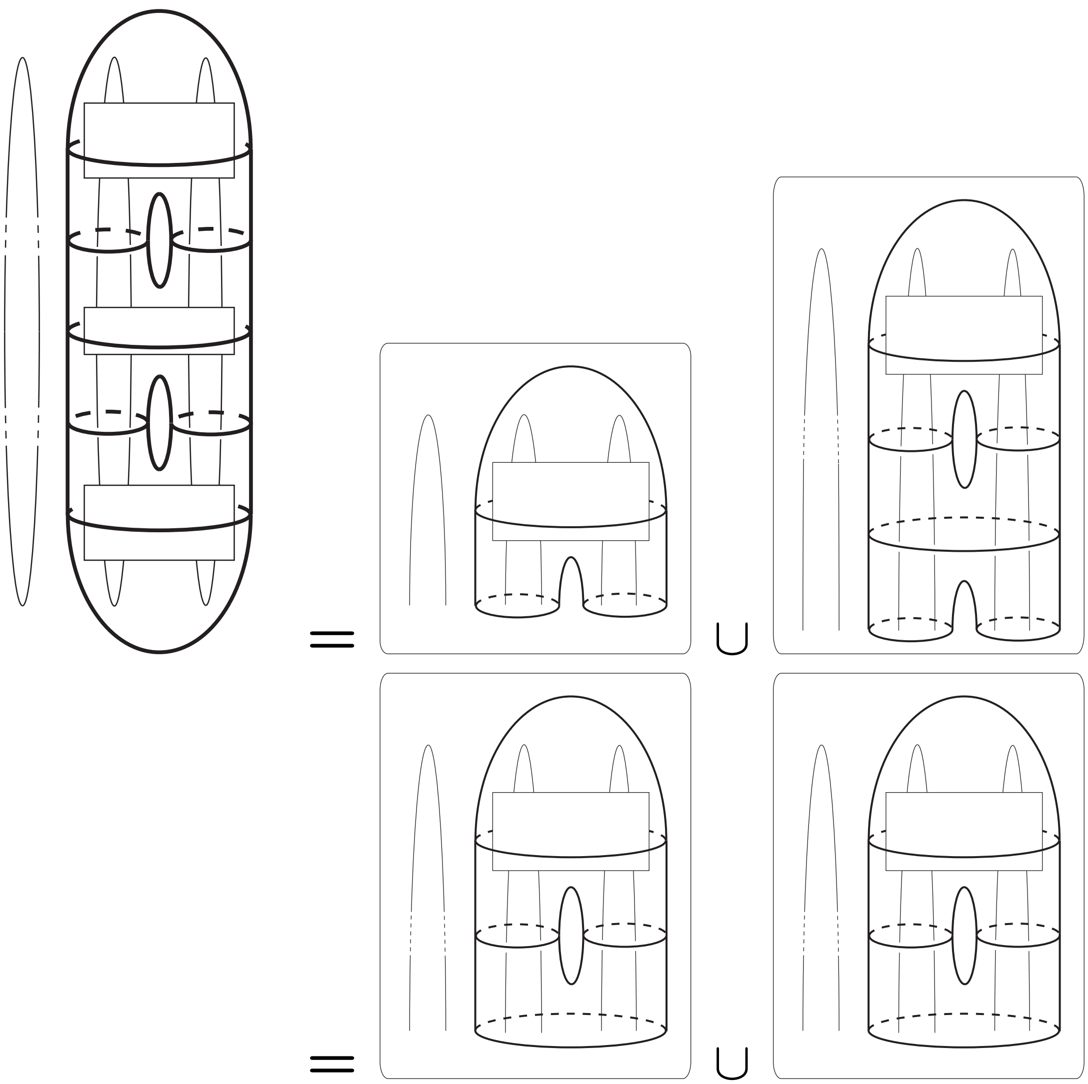
torus(0)-2



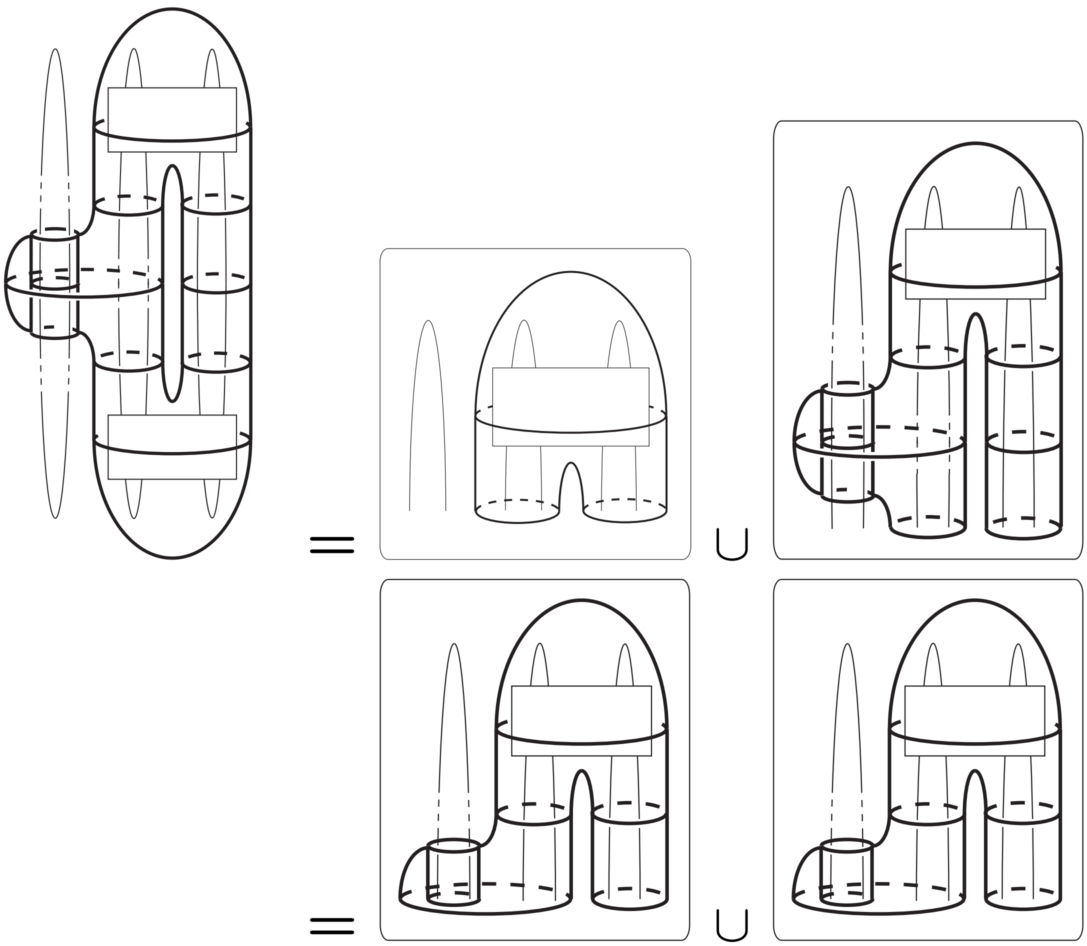
torus(0)-3

By Lemma A and the classification of incompressible and meridionally incompressible surfaces in the trivial 3-string tangle, we obtain Theorem I, II and III.

I-a



I–b



I–c