Rational structure on algebraic tangles and closed incompressible surfaces in the complements of algebraically alternating knots and links

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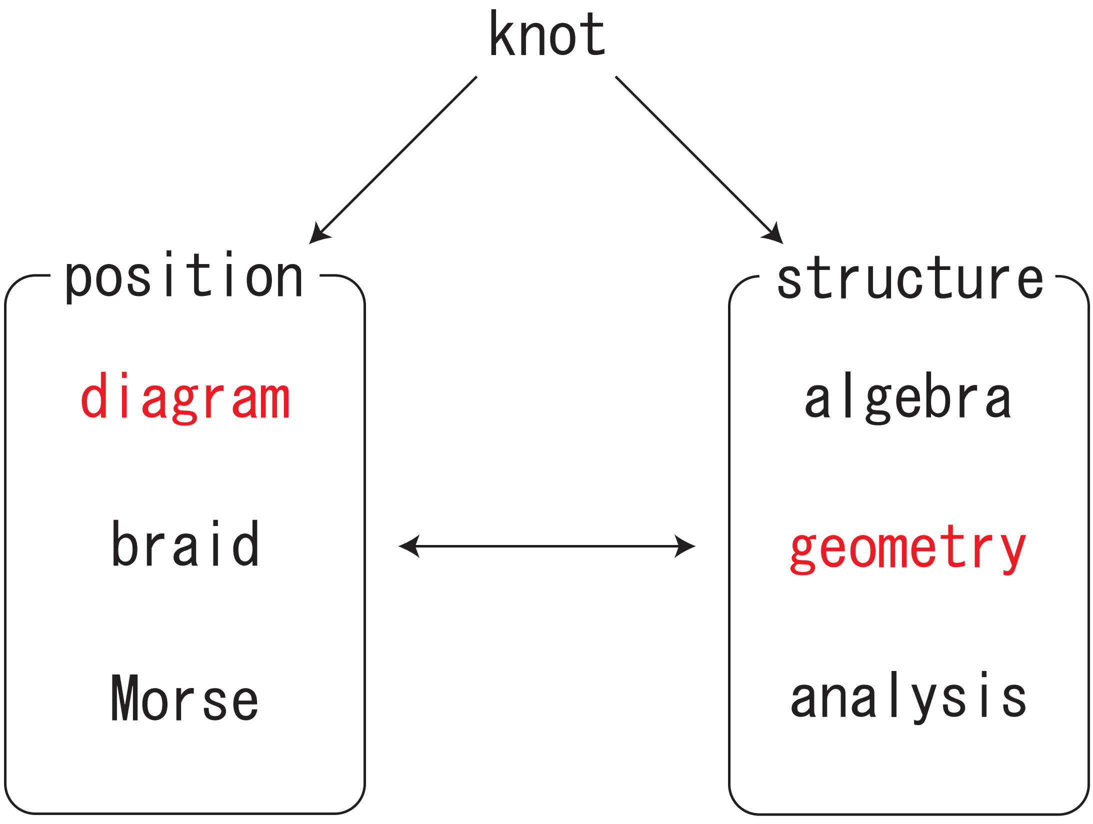
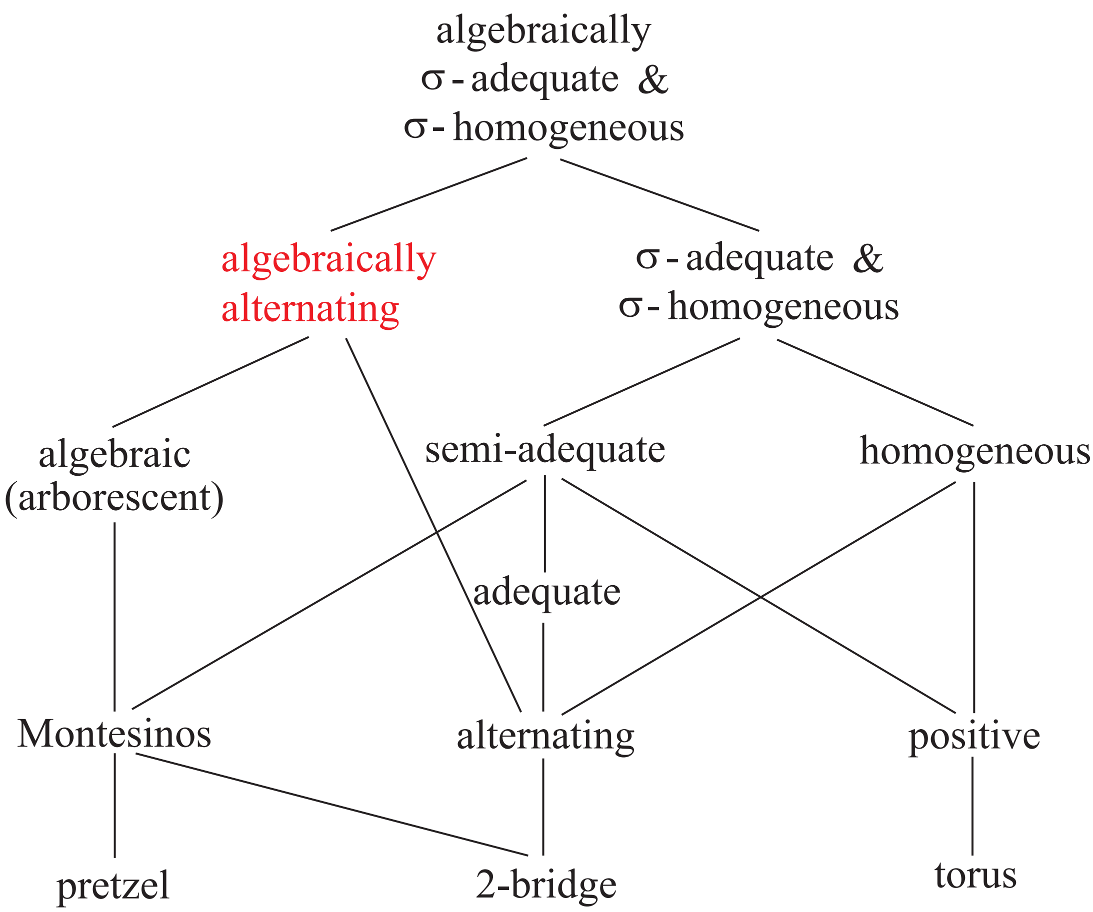
How can we recognize knots?



この講演では、diagram（algebraically alternating）と geometry（closed incompressible surface）との関連性についてお話します。

The Hasse diagram for knot diagrams



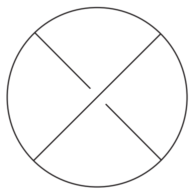
algebraically alternatingは、algebraicとalternatingを含むクラスです。

Essential surfaces in knot/link exteriors

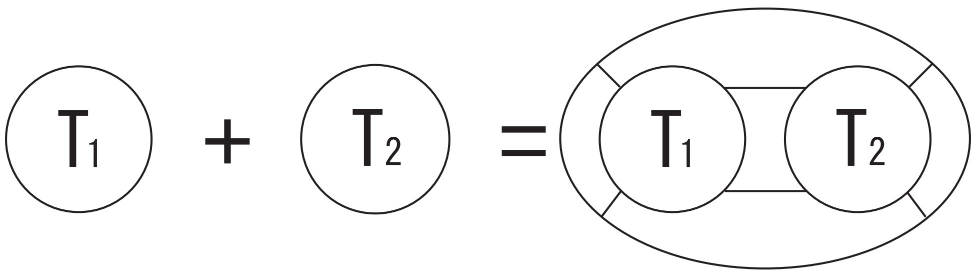
- $\partial = \emptyset$
  - **Closed surface**
    - $g = 0 \iff$ split link
    - $g = 1$, non-separating
      $\iff$ satellite knot/link

- $\partial \neq \emptyset$
  - **Seifert surface**
    - $g = 0$, $|\partial| = 1 \iff$ trivial knot
  - **Tangle decomposing sphere**
    - $|\partial| = 2 \iff$ composite knot/link
  - **Interpolating surface**
    - $g = 1 \iff$ torus or cable knot/link

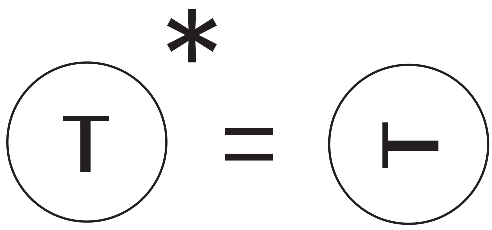
Algebraic tangles



1/1-tangle



tangle sum



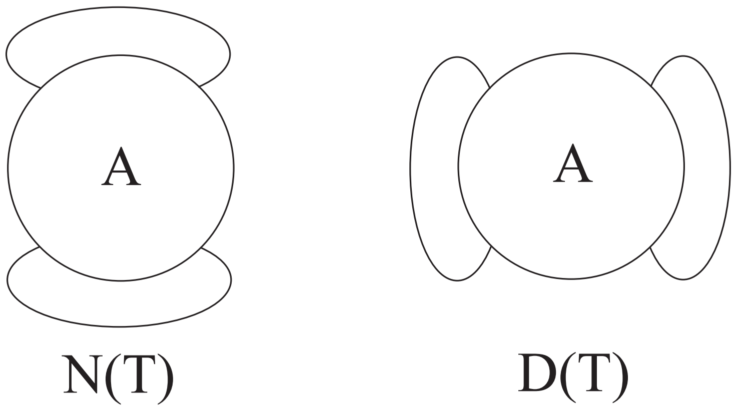
rotation

algebraic tangle は、1/1-tangle から、tangle sum と rotation によって得られる tangle です。

Formulae on tangle fractions

T : tangle
$p = det(N(T))$
$q = det(D(T))$

とするとき、$f(T) = p/q$ を **tangle fraction** という。



Theorem (Conway)

$f(T_i) = p_i/q_i \ (i = 1, 2)$ とするとき、

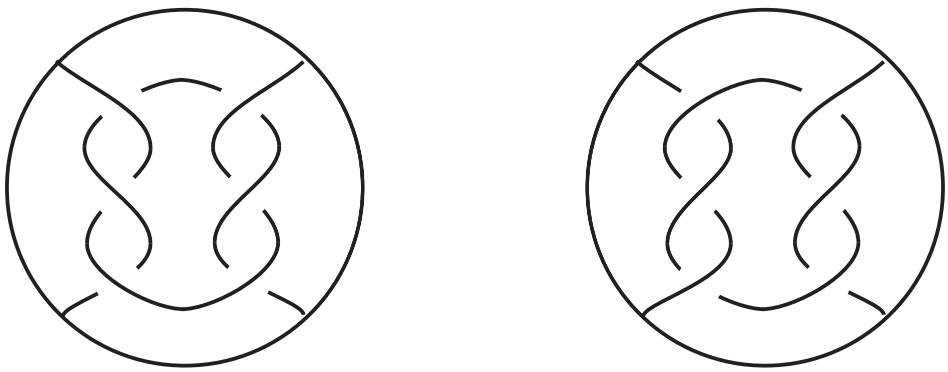
- $f(T_1 + T_2) = \dfrac{p_1q_2 + p_2q_1}{q_1q_2}$
- $f(T_1^*) = -\dfrac{1}{f(T_1)}$

Theorem (Krebes)

$(B, T) \subset (S^3, K)$

$f(T) = p/q$

$\Rightarrow gcd(p, q) | det(K)$



Square tangle T_1 Granny tangle T_2

$$f(T_1) = \frac{-1}{3} + \frac{1}{3} = \frac{0}{9} \quad f(T_2) = \frac{1}{3} + \frac{1}{3} = \frac{6}{9}$$

$$gcd(0, 9) = 9 \quad gcd(6, 9) = 3$$

$$T_1 \not\subset 0_1, \ 3_1, \ etc. \quad T_2 \not\subset 0_1, \ 2_1^2, \ etc.$$

Remark. $N(T_2 + -1/1) = 3_1$

m-essential surfaces

surface F が**essential**であるとは、

- incompressible
- ∂-incompressible
- not ∂-parallel

のときをいう。

更に、

- meridionally incompressible

のとき、**m-essential**という。

Theorem (Menasco)

K : alternating knot/link

$\Rightarrow \not\exists F \subset S^3 - K$: m-essential closed surface

Theorem (Oertel)

K : Montesinos knot/link

$\exists F \subset S^3 - K$: m-essential torus

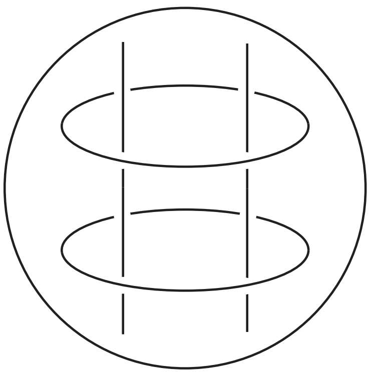
$\iff K = P(p, q, r, -1), \ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$

Theorem (Bonahon-Siebenmann)

K : large arborescent knot/link

$\exists F \subset S^3 - K$: m-essential torus

$\iff \exists Q_2 \subset (S^3, K)$



Q_2

m-essential surfaces in algebraic tangles

$\mathcal{E}(T)$: the set of m-**E**ssential surfaces
$\mathcal{B}(T)$: the set of m-essential surfaces with **B**oundary
$\mathcal{C}(T)$: the set of m-essential **C**losed surfaces
Then, $\mathcal{E}(T) = \mathcal{B}(T) \cup \mathcal{C}(T)$

T has **Property** ∞
$\iff$ T contains a sub-tangle which is obtained by a tangle sum of slope $1/0 + 1/0$

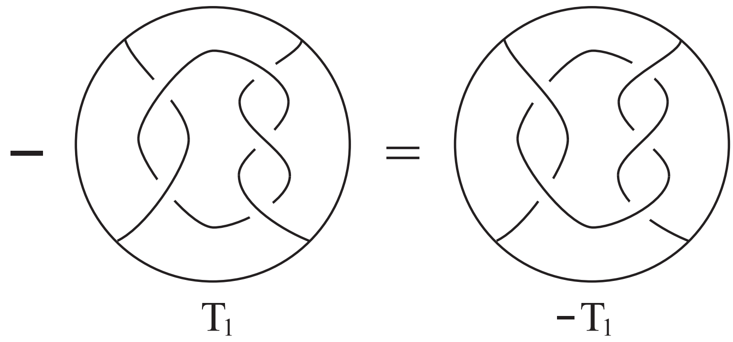
Theorem 1

T : algebraic tangle

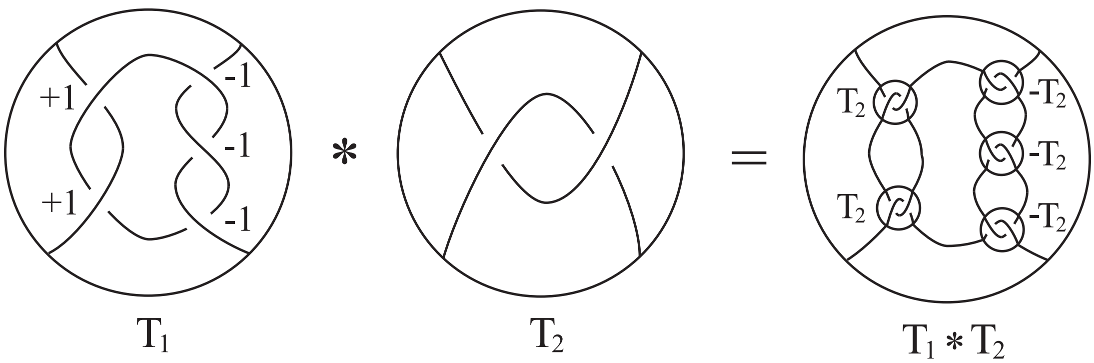
1. $\forall T$, $\exists F \in \mathcal{B}(T)$: free, separating
2. $\forall F \in \mathcal{B}(T)$ has a unique ∂-slope
3. $|\mathcal{B}(T)| = 1 \iff \mathcal{C}(T) = \emptyset$
   $\iff$ T does not have Property ∞

Tの∂-slope及びgenusを
$\phi(T) = \partial$-slope of $F \in \mathcal{B}(T)$
$g(T) = \min\{g(F) | F \in \mathcal{B}(T)\}$
とおく。

Formulae on tangle slopes



reflection



tangle multiplication

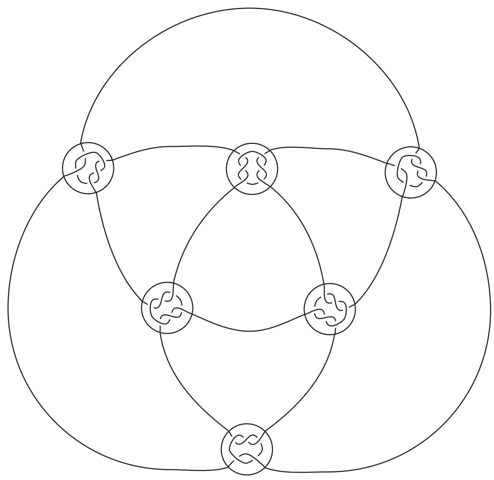
Theorem 2

T_1, T_2: algebraic tangles

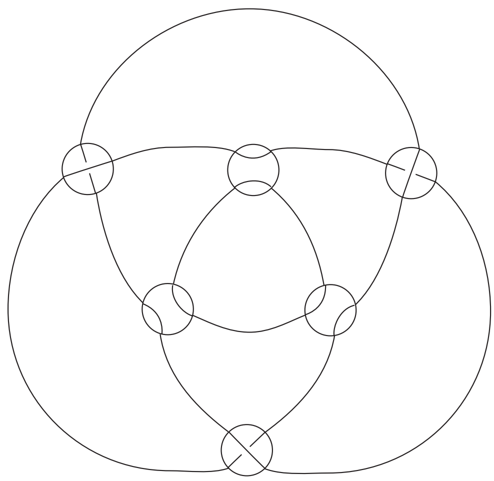
1. $\phi(T_1 + T_2) = \phi(T_1) + \phi(T_2)$
2. $\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$
3. $\phi(-T_1) = -\phi(T_1)$
4. $\phi(T_1^*) = -\frac{1}{\phi(T_1)}$

Algebraically alternating diagrams

Conway notation において、各 algebraic tangle T を、$\phi(T) > 0$, < 0, 0, ∞ ならば、それぞれ $1/1$, $-1/1$, 0, ∞ の rational tangle に置き換えた diagram を **basic diagram** という。



$\tilde{K}$: algebraically alternating link diagram



$\tilde{K}_0$: the basic diagram of $\tilde{K}$

m-essential closed surfaces in the complements of algebraically alternating knots and links

Theorem 3

K : algebraically alternating knot/link in S^3

$\tilde{K}$: algebraically alternating diagram of K

$\tilde{K}_0$: basic diagram of $\tilde{K}$

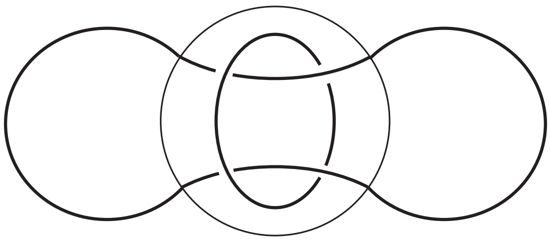
1. $\exists F \subset S^3 - K$: m-essential closed surface $\iff$ $\tilde{K}_0$ is split or $\exists T$: an algebraic tangle of $\tilde{K}$ with Property ∞
2. $\exists S^2 \subset S^3 - K$: m-essential 2-sphere $\iff$ $\exists T$: $g = 0$ cut tangle
3. Suppose $\not\exists T$: $g = 0$ cut tangle
   Then $\exists T^2 \subset S^3 - K$: m-essential torus $\iff$ $\exists T$: $g = 1$ cut tangle or $\exists Q_2 \subset (S^3, K)$

Corollary

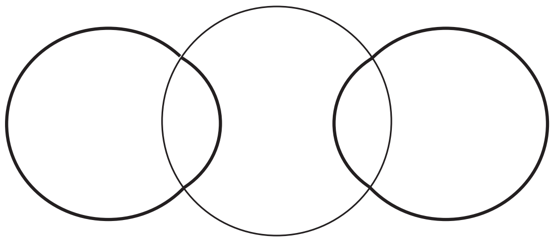
K : algebraically alternating knot

$\Rightarrow \not\exists F \subset S^3 - K$: m-essential closed surface

Example. $\exists F \subset S^3 - K$: m-essential sphere, and $\exists g = 0$ cut tangle

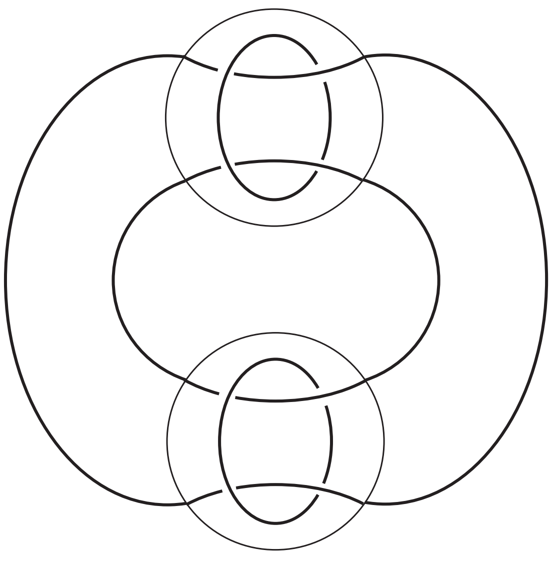


$\tilde{K}$

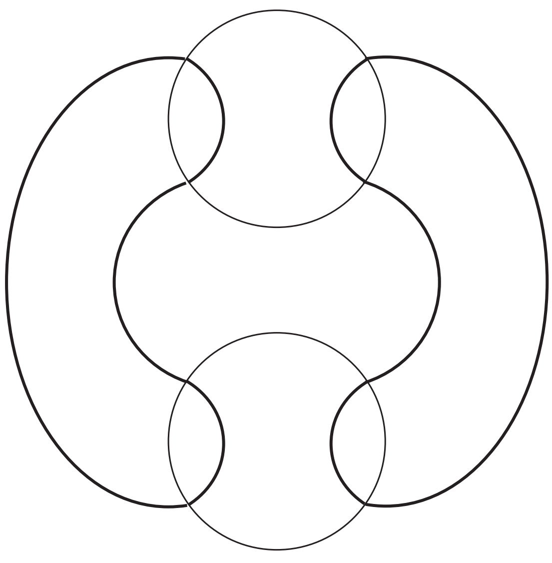


$\tilde{K}_0$

Example. $\exists F \subset S^3 - K$: m-essential torus, and $\exists Q_2 \subset (S^3, K)$



$\tilde{K}$



$\tilde{K}_0$

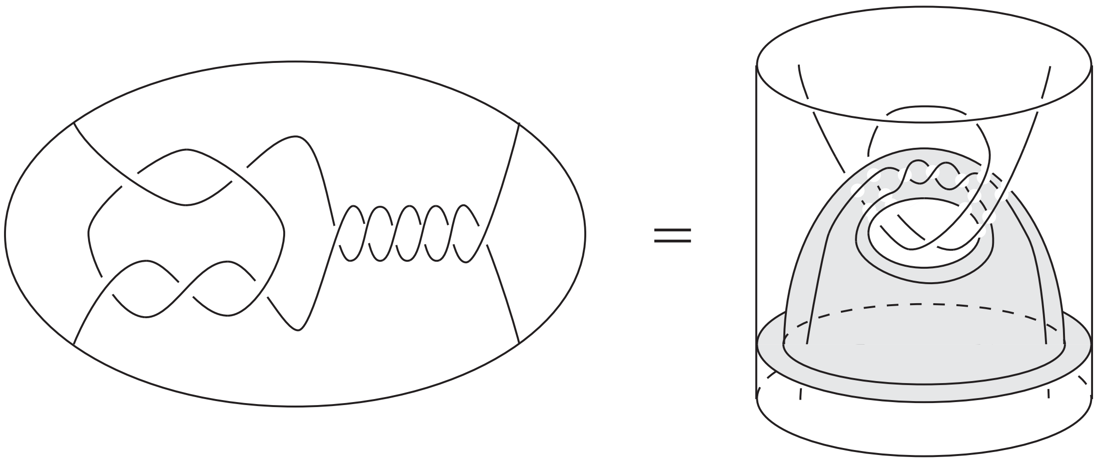
Example.

$N(T) = P(2, -3, -6)$

$\exists F \subset S^3 - N(T)$: m-essential torus,

but $\nexists Q_2 \subset (S^3, K)$,

because T is a $g = 1$ cut tangle in $N(T)$



$T = R[1/2, -1/3; -1/6]$

Problem.
$\forall T$: 2-string tangle
$\exists^{?} F \subset B^3 - T$: m-essential surface with boundary, and are ∂-slopes of $F \in \mathcal{B}(T)$ unique?

Problem.
Characterize $g = 0$, 1 cut tangles

Problem.
Classify algebraically alternating links

Problem.
$K \leftrightarrow K'$
$\iff \exists (B, T) \subset (S^3, K)$, $\exists (B', T') \subset (S^3, K')$, s.t. $(S^3 - B, K - T) \simeq (S^3 - B', K' - T')$ and $\phi(T) = \phi(T')$

$K \sim K_n$
$\iff \exists K = K_0,\ K_1,\ \dots,\ K_n$ s.t. $K_i \leftrightarrow K_{i+1}$ $(i = 0,\ 1,\ \dots,\ n-1)$

What can we say about this equivalent relation?