PRIMITIVE SPATIAL GRAPHS AND GRAPH MINORS

MAKOTO OZAWA AND YUKIHIRO TSUTSUMI

Abstract. Robertson, Seymour and Thomas characterized linkless embeddings of graphs by flat embeddings, and determined the obstruction set for linkless embeddings. In this paper, we extend flat embeddings to "primitive embeddings" as linkless embeddings to knotless embeddings. Although the obstruction set for knotless embeddings has not been determined, fundamental theorems and conjectures are obtained.

1. Introduction

Let G be a finite graph. An embedding of G into the 3-sphere S^3 is said to be *knotless* (resp. *linkless*) if it contains no non-trivial knot (resp. non-trivial link) from cycles of G. Let us say that an embedding ϕ of G in S^3 is *free* if the fundamental group of $S^3 - \phi(G)$ is free. An embedding ϕ of G in S^3 is said to be *flat* if for every cycle C of G, there exists a disk in S^3 internally disjoint from $\phi(G)$ whose boundary is $\phi(C)$. The *Petersen family* is the set of all graphs that are obtained from K_6 or $K_{3,3,1}$ by a finite sequence of ΔY-exchanges. We remark that the Petersen family results in the set of all graphs that can be obtained from K_6 by means of $Y\Delta$- and ΔY-exchanges. See Figure 5 in Section 2 for $Y\Delta$- and ΔY-exchanges and Figure 1 in [15] for the Petersen family. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Robertson, Seymour and Thomas ([13]) proved that the followings are equivalent.

1. G has an embedding whose every subgraph is free,
2. G has a flat embedding,
3. G has a linkless embedding,
4. G has no minor in the Petersen family.

1.1. Fundamental Theorem and Conjecture.

For knotless embeddings, such characterization has yet to be done. Here, we introduce a notion of primitive embeddings, which will make a contribution to the knotless embedding problem. An embedding ϕ of a graph G in S^3 is said to be *primitive* if for each component G_i of G and any spanning tree T_i of G_i, the bouquet $\phi(G_i)/\phi(T_i)$ obtained from $\phi(G_i)$ by contracting all edges of $\phi(T_i)$ in S^3 is trivial.

Theorem 1.1. *An embedding of a graph in the 3-sphere is primitive if and only if the restriction on its any connected subgraph is free.*

Therefore, a flat embedding is primitive, but the converse is not true. For example, a non-split spatial graph that consists of trivial disjoint cycles is primitive, but

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not flat. Besides, a primitive embedding is knotless since the fundamental group of a non-trivial knot complement is not free. As we know afterward, the notion of primitive embedding is closely related to knotless embedding. We expect;

Conjecture 1.2. A graph has a primitive embedding if and only if it has a knotless embedding.

Remark 1.3. Minimally knotted embeddings of a connected graph are knotless, but not primitive since itself is not free by [11].

The following theorem assures us of a relation about edge-deletions and contractions (compare with (1.8) in [16] and an alternate version of 7.5 in [18]).

Theorem 1.4. *Let ϕ be an embedding of a graph G, and e be an edge of G with distinct end vertices. Then ϕ is primitive (resp. knotless) if and only if both of $\phi|_{G-e}$ and $\phi|_{G/e}$ are primitive (resp. knotless).*

For a $\Delta Y/Y\Delta$-exchange, we have a relation between two embeddings (compare with Lemma in [10]).

Theorem 1.5. *Let H be a graph with a 3-cycle C, and ϕ' be an embedding of H such that C bounds a disk D internally disjoint from $\phi'(H)$. Let G be a graph obtained from H by a ΔY-exchange, and ϕ be an embedding of G which is obtained from ϕ' by a ΔY-exchange on D. Then ϕ is primitive (resp. knotless) if and only if ϕ' is primitive (resp. knotless).*

1.2. **Graph minor.** For graph minors, primitive embeddings are similar to knotless embeddings. An abstract graph is *primitive* if it has a primitive embedding.

Theorem 1.6. *The property of having a primitive embedding is preserved under taking minors.*

Let $\mathcal{C}$ be a property closed under minor-reduction. The *obstruction set* for $\mathcal{C}$, denoted by $\Omega(\mathcal{C})$, is the set of all minor-minimal graphs which do not have $\mathcal{C}$. It is well-known that $\Omega(\mathcal{C})$ is finite ([12]). Therefore, it is a characterization for a property $\mathcal{C}$ to determine the obstruction set. Let $\mathcal{KL}$ be the property that a graph has a knotless embedding. Kohara and Suzuki ([10]) conjectured that $\Omega(\mathcal{KL})$ is equal to the union of K_7-family and $K_{3,3,1,1}$-family, where K_7*-family* (resp. $K_{3,3,1,1}$-*family*) is the set of graphs that are obtained from K_7 (resp. $K_{3,3,1,1}$) by ΔY-exchanges. Recently, Foisy ([4]) discovered a new intrinsically knotted graph, which we call Foisy graph and denote by F, belongs to $\Omega(\mathcal{KL})$, but is independent of the K_7- and $K_{3,3,1,1}$-family. Let $\mathcal{P}$ be the property that a graph has a primitive embedding.

Theorem 1.7. *The K_7-family and $K_{3,3,1,1}$-family are contained in the obstruction set $\Omega(\mathcal{P})$ for primitive graphs.*

Remark 1.8. Since the Foisy graph F is not primitive, there exists a graph $G \in \Omega(P)$ which is a minor of F. This graph G will be a new element of $\Omega(P)$ other than K_7 and $K_{3,3,1,1}$-family, since F has no minor in K_7 nor $K_{3,3,1,1}$-family.

It is clear that a graph obtained from a planar graph joined with two vertices has a knotless embedding. Indeed, when we construct an embedding of the graph forming a plane graph joined with the North Pole and the South Pole, any cycle is a bridge number one knot, hence unknotted.

Theorem 1.9. *A planar graph joined with two vertices is primitive.*

Remark 1.10. For any graph G in K_7- or $K_{3,3,1,1}$-family, any minor of G forms a planar graph joined with two vertices. However, there is an edge e of the Foisy graph F such that $F - e$ is not a form of planar graph joined with two vertices.

1.3. **Primitive embedding.** By Theorem 1.1 and the Robertson-Seymour-Thomas theorem, flat embeddings are primitve. Conversely, primitive embeddings are flat if we restrict on the structure of graphs.

Theorem 1.11. *If a graph has no disjoint cycles, then any primitive embedding of the graph is also flat.*

If a graph has disjoint cycles, then primitive embeddings of the graph are not flat generally. We characterized primitive embeddings of a "handcuff graph with n-bridges" H_n for $n = 1, 2, 3$.

A link L is called a 2-*bridge link* if there is a sphere which intersects L transversely in four points and decomposes (S^3, L) into two trivial 2-string tangles. A link L is called a (p, q)-*torus link* if there is a solid torus V standardly embedded in S^3 so that L is contained in ∂V as a (p, q)-curve, where $(0, 1)$ and $(1, 0)$ correspond to a meridian and a preferred longitude for V respectively. An unknotting tunnel τ is an arc such that $\tau \cap L = \partial \tau$ and $S^3 - intN(L \cup \tau)$ is a handlebody. We note that every 2-bridge link and torus link admits an unknotting tunnel.

We recall that a complete classification of the unknotting tunnels for 2-bridge links is given by Adams and and Reid ([1]), and that only the upper and lower tunnels are unknotting tunnels. See [8] for the classification of unknotting tunnels of 2-bridge knots.

Theorem 1.12. *Any primitive embedding of H_n $(n = 1, 2, 3)$ forms :*

1. *a 2-bridge link with an upper tunnel if* $n = 1$.
2. *a 2-bridge link with an upper tunnel and a lower tunnel if* $n = 2$.
3. *a $(2, q)$-torus link with three parallel tunnels if* $n = 3$.

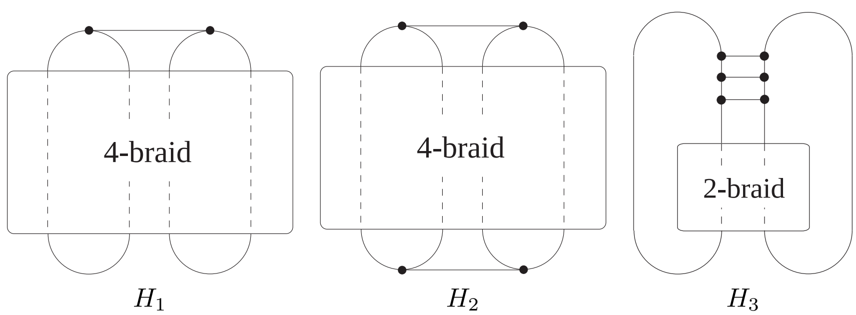

FIGURE 1. Primitive embedding of H_n $(n = 1, 2, 3)$

From Theorem 1.12, we know that primitive embeddings are "rigid" as the connectivity of the graph becomes higher. It should be noted that any link contained in any primitive embedding of K_6 is a $(2, q)$-torus link since K_6 contains H_3 as a primitive embedding by Theorem 1.4. In this direction, we present a non-planar graph which has exactly two primitive embeddings up to reflection, as follows.

Let K_5' be a graph obtained from K_5 by once de-contracting, that is, there exists an edge e of K_5' such that K_5'/e is isomorphic to K_5. See Figure 2.

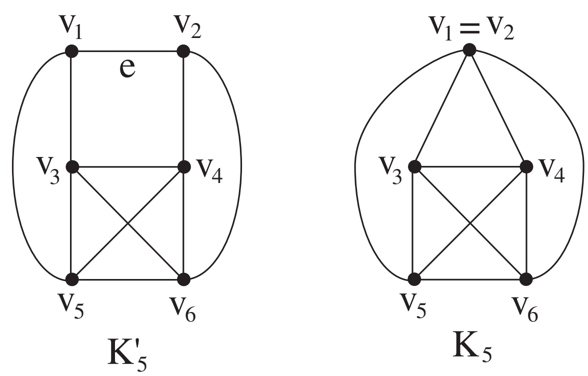

FIGURE 2. K_5' and K_5

Lemma 1.13. *K_5' has exactly two primitive embeddings up to reflection, as illustrated in Figure 3.*

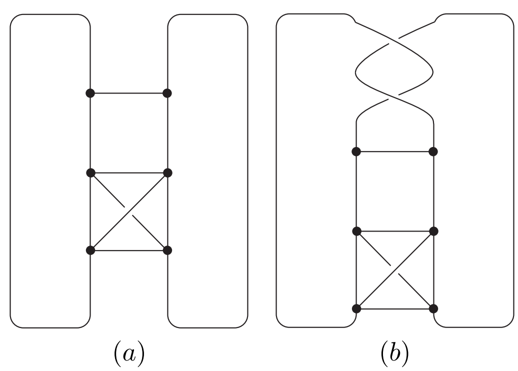

FIGURE 3. two primitive embeddings of K_5'

Remark 1.14. It follows from Lemma 1.13 that the only pair of disjoint cycles $v_1v_3v_5v_1$ and $v_2v_4v_6v_2$ of K_5' forms the trivial link or the Hopf link in any primitive embedding of K_5'.

By using Lemma 1.13, we can show a strong restriction on sublinks contained in a primitive embedding of a graph in the Petersen family.

Theorem 1.15. *Any link contained in a primitive embedding of a graph in the Petersen family is either the trivial link or the Hopf link.*

Generally, sublinks contained in a primitive embedding are under restriction.

Theorem 1.16. *An n-component link contained in a primitive embedding of a connected graph has bridge number n.*

According to (1.7) in [16], flat embeddings of a 4-connected graph are unique up to reflections. We also expect some rigidity of primitive embeddings.

Conjecture 1.17. Primitive embeddings of a 5-connected graph are unique up to reflections.

For planar graphs, by Theorem 1.11, any primitive embedding of a planar graph is flat if it has no disjoint cycles, hence primitive embeddings of such graphs are unique. This is the only case for planar graphs to have a unique primitive embedding.

Theorem 1.18. *A planar graph has a unique primitive embedding if and only if it has no disjoint cycles. Moreover, if a planar graph has disjoint cycles, then it has infinitely many primitive embeddings.*

2. Proof of Fundamental Theorems

We refer the reader to [17], [6], [7] for standard terminology in knot theory and three-dimensional topology. A *tangle* is a pair of a 3-ball B and properly embedded 1-manifold t. When t consists of n arcs, we call the tangle an n-string tangle. An n-string tangle is said to be *trivial* if there are mutually disjoint n disks D_i's in B such that $\partial B \cap D_i$ is a subarc of ∂D_i, $D_i \cap t = t_i$, and $\partial D_i = t_i \cup (B \cap) - \partial B \cap D_i$, where t_i is a component of t.

Let $\phi : G \to S^3$ be an embedding of G, and e an edge of G. A simple closed curve on $\partial N(\phi(G))$ is called a *meridian for* $\phi(e)$ if it bounds a disk D in $N(\phi(G))$ such that $D \cap \phi(G)$ is a single transverse point, where $N(\phi(G))$ is a regular neighborhood of $\phi(G)$. We denote by $\phi(e)^*$ the meridian for $\phi(e)$, and call such a disk D a *meridian disk for* $\phi(e)$. For a subgraph G' of G, there are disjoint meridian disks for the edges of G' and we denote by $\phi(G')^*$ the disjoint union of meridians for all edges of G'.

If we contract $\phi(e)$ in S^3, then the resultant embedding of G/e, denoted by the same symbol ϕ or $\phi|_{G/e}$, is well-defined as $\phi(G/e) = \phi(G)/\phi(e)$. Notice that $(N(\phi(G/e)), \phi(G/e)^*, S^3)$ is homeomorphic to $(N(\phi(G)), \phi(G)^* - \phi(e)^*, S^3)$.

Let T be a spanning tree of a connected graph G. We call the set of the edges $\mathcal{E}_T = E(G) - E(T)$ a *base edge system* for T. A set of edges $\mathcal{E}$ of G is called a *base edge system* if there is a spanning tree T such that $E(G) - E(T) = \mathcal{E}$.

If we contract all edges of $\phi(T)$ in S^3, then the resultant embedding of G/T is uniquely defined. We denote this embedding by the same symbol ϕ. Thus we regard $\phi(G)/\phi(T)$ as $\phi(G/T)$.

Lemma 2.1. *An embedding ϕ of G is primitive if and only if for any base edge system $\mathcal{E} = \{e_1, \dots, e_n\}$ there exist disks $D_1, \dots, D_n$ in $S^3 - intN(\phi(G))$ such that $|\partial D_i \cap \phi(e_j)^*| = \delta^i_j$ $(i, j = 1, \dots, n)$.*

Proof. Let T be a spanning tree for G and $\mathcal{E}_T = \{e_1, \dots, e_n\}$ a base edge system for T. Put $B = S^3 - intN(\phi(T))$ and $B' = S^3 - intN(\phi(T/T))$. Then, ϕ is primitive if and only if the n-string tangle $(B', \phi(e_1 \cup \dots \cup e_n) \cap B')$ is trivial. On the other hand, the latter condition of Lemma 2.1 holds if and only if the n-string tangle $(B, \phi(e_1 \cup \dots \cup e_n) \cap B)$ is trivial. Hence, Lemma 2.1 is proved since these two tangles are equivalent. □

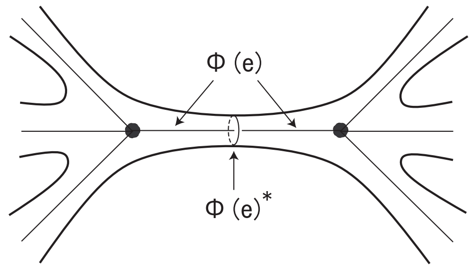

FIGURE 4. meridian $\phi(e)^*$ for $\phi(e)$

Remark 2.2. When G is connected, $\phi : G \to S^3$ is primitive if and only if $S^3 - intN(\phi(G))$ is a handlebody and $\mathcal{E}_T^*$ is a primitive set on $\partial(S^3 - intN(\phi(G)))$ in the sense of Gordon [5] for any spanning tree T.

Proof. (of Theorem 1.1) Suppose that an embedding ϕ of G is primitive. Let G' be a connected subgraph of G and T' a spanning tree for G'. We extend T' to a spanning tree T for G. Since G is primitive, $\phi(G)/\phi(T)$ is a trivial bouquet. Therefore $\phi(G')/\phi(T') \subset \phi(G)/\phi(T)$ is also a trivial bouquet. Since $\pi_1(S^3 - \phi(G'))$ is homomorphic to $\pi_1(S^3 - \phi(G')/\phi(T'))$, $\phi(G')$ is free.

Conversely, suppose that for any connected subgraph G' of G, $\phi(G')$ is free. Let T be a spanning tree for G and $\mathcal{E}$ the base edge system for T. Then $B = S^3 - intN(\phi(T))$ is a 3-ball, and by the supposition, $B - intN(\phi(\mathcal{E}'))$ is a handlebody for all subsets $\mathcal{E}'$ of $\mathcal{E}$. By Theorem 1 in [5] and Lemma 2.1, ϕ is primitive. □

Proof. (of Theorem 1.4) [Primitive] Suppose that ϕ is primitive, and let H be a connected subgraph of $G - e$. Then $\phi(H)$ is free since H is a connected subgraph of G. Hence $\phi|_{G-e}$ is primitive. Next, let J be a connected subgraph of G/e, and J' be the corresponding subgraph of G. Then $\phi(J)$ is free since $\pi_1(S^3 - \phi(J)) \cong \pi_1(S^3 - \phi(J'))$. Hence $\phi|_{G/e}$ is primitive.

Conversely, suppose that both of $\phi|_{G-e}$ and $\phi|_{G/e}$ are primitive, and let H be a connected subgraph of G. If H contains e, then $\phi(H)$ is free since $\pi_1(S^3 - \phi(H)) \cong \pi_1(S^3 - \phi(H/e))$ and $\phi(H/e)$ is free. Otherwise, $\phi(H)$ is free since H is a connected subgraph of $G - e$. Hence ϕ is primitive.

[Knotless] Suppose that ϕ is knotless. If $\phi(G)/\phi(e)$ contains a non-trivial knot K, then K passes through the vertex $\phi(e)/\phi(e)$ just once. Hence, K is also contained in $\phi(G)$, and this contradicts that ϕ is knotless. It is easy to see that $\phi(G-e)$ is knotless since $G - e$ is a subgraph of G. Conversely, suppose that both of $\phi|_{G-e}$ and $\phi|_{G/e}$ are knotless, and that $\phi(G)$ contains a non-trivial knot K. If K contains e, then it is contained in $\phi(G)/\phi(e)$. This contradicts that $\phi|_{G/e}$ is knotless. Otherwise, K is contained in $\phi(G - e)$, and we have a contradiciton in the same way. Hence, ϕ is knotless. □

Before proving Theorem 1.5, we need to prepare some lemmas.

Lemma 2.3. *Let G be a graph with a valency 2 vertex v to which two edges e_1 and e_2 incident, and ϕ be an embedding of G. Then ϕ is primitive (resp. knotless) if and only if $\phi|_{G/e_1}$ is primitive (resp. knotless).*

Proof. Suppose that ϕ is primitive. Then $\phi(G)/\phi(e_1)$ is primitive by Theorem 1.4.

Conversely, suppose that $\phi|_{G/e_1}$ is primitive. By Theorem 1.4, $\phi|_{(G-e_1)/e_2}$ is primitive since $\phi(G-e_1)/\phi(e_2) = \phi(G)/\phi(e_1) - \phi(e_2)$. And $\phi|_{(G-e_1)-e_2}$ is primitive since $\phi(G-e_1)-\phi(e_2) = (\phi(G)/\phi(e_1)-\phi(e_2)) \cup$ (isolated vertex). Hence, by Theorem 1.4, $\phi|_{G-e_1}$ is primitive. Therefore, $\phi|_G$ is primitive. □

Lemma 2.4. *Let G be a graph with a loop e, and ϕ be an embedding of G such that $\phi(e)$ bounds a disk internally disjoint from $\phi(G)$. Then ϕ is primitive (resp. knotless) if and only if $\phi|_{G-e}$ is primitive (resp. knotless).*

Proof. Suppose that ϕ is primitive. Then $\phi(G-e)$ is primitive by Theorem 1.4.

Conversely, suppose that $\phi|_{G-e}$ is primitive. Let T be a spanning tree of G. Then $\phi(G)/\phi(T)$ is a trivial bouquet since $\phi(G)/\phi(T) = (\phi(G-e)/\phi(T)) \cup$ (trivial loop). Hence ϕ is primitive. □

Lemma 2.5. *Let G be a graph with multi-edges e_1 and e_2, and ϕ be an embedding of G such that $\phi(e_1 \cup e_2)$ bounds a disk internally disjoint from $\phi(G)$. Then ϕ is primitive (resp. knotless) if and only if $\phi|_{G-e_1}$ is primitive (resp. knotless).*

Proof. Suppose that ϕ is primitive. Then $\phi(G-e_1)$ is primitive by Theorem 1.4.

Conversely, suppose that $\phi|_{G-e_1}$ is primitive. By Theorem 1.4, $\phi|_{G/e_1-e_2}$ is primitive since $\phi(G)/\phi(e_1)-\phi(e_2) = \phi(G-e_1)/\phi(e_2)$. Therefore by Lemma 2.4, ϕ_{G/e_1} is primitive since $\phi(e_2)$ bounds a disk internally disjoint from $\phi(G)/\phi(e_1)$. Hence, by Theorem 1.4, ϕ is primitive. □

Let v be a vertex of a graph G of valency three with distinct neighbors v_1, v_2, v_3. Let H be obtained from G by deleting v and adding edges e_1, e_2, e_3, where $e_1 = v_2v_3, e_2 = v_3v_1, e_3 = v_1v_2$. We say that H is obtained from G by a *$Y\Delta$-exchange* and that G is obtained from H by a *ΔY-exchange*.

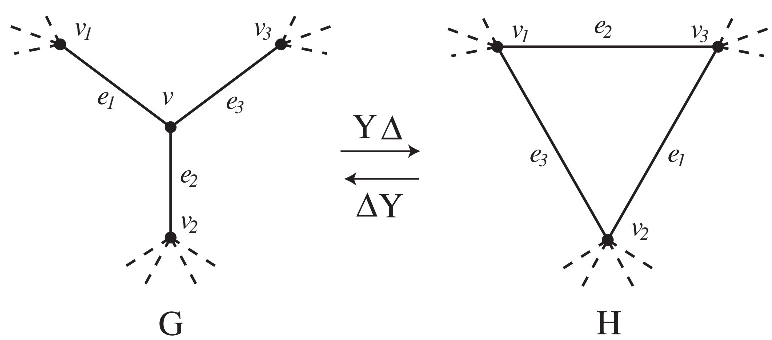

FIGURE 5. $Y\Delta$- and ΔY-exchange

Proof. (of Theorem 1.5) We use the labelling in Figure 5.

[Primitive] Suppose that ϕ is primitive. Then $\phi'(H/e_1)-\phi'(e_3)$ is primitive since $\phi'(H/e_1)-\phi'(e_3) = \phi(G/(e_2 \cup e_3))$. Therefore by Lemma 2.5, $\phi'(H/e_1)$ is primitive since $\phi'(e_2 \cup e_3)$ bounds a disk internally disjoint from $\phi'(H/e_1)$. And $\phi'(H-e_1)$is primitive since $\phi'(H-e_1) = \phi(G/e_1)$. Hence by Theorem 1.4, ϕ' is primitive.

Conversely, suppose that ϕ' is primitive. Then $\phi(G/e_1)$ is primitive since $\phi(G/e_1) = \phi'(H-e_1)$. And $\phi(G-e_1)/\phi(e_2)$ is primitive since $\phi(G-e_1)/\phi(e_2) =$

$\phi'(H) - \phi'(e_2 \cup e_3)$. Therefore by Lemma 2.3, $\phi(G - e_1)$ is primitive. Hence by Theorem 1.4, ϕ is primitive.

[Knotless] The proof is similar to above, we need to use only Theorem 1.4, Lemma 2.3, 2.4, 2.5. □

3. Proofs of Theorems on graph minor

Lemma 3.1. *Let $\mathcal{C}$ be a property preserved under taking minors, multiplication of edges, adding loops, and $Y\Delta$-exchanges. Let H be a graph obtained from G by a ΔY-exchange. Suppose that G does not have $\mathcal{C}$ and suppose that H is a forbidden graph for $\mathcal{C}$. Then G is also a forbidden graph for $\mathcal{C}$.*

Proof. Let e be an edge of G. It is sufficient to show that $G - e$ and G/e have $\mathcal{C}$.

Regarding the triangle of G, we have the following three cases: (A): $e = e_1$, (B): $\partial e = v_1 \cup v_2$ and $e \neq e_3$, and (C): otherwise.

(A): In this case, $G - e = H/v_1v_4$ and thus $G - e$ has $\mathcal{C}$. On the other hand, G/e is obtained from $H/\{v_2v, v_3v\}$ by adding a parallel edge to v_1v. Hence G/e has $\mathcal{C}$. (B): In this case, $G - e$ is obtained from $H - e$ by a $Y\Delta$-exchange. Thus, $G - e$ has $\mathcal{C}$. On the other hand, G/e is obtained from $H/\{v_1v, v_2v\}$ by adding a loop to v. Hence G/e has $\mathcal{C}$. (C): In this case, $G - e$ and G/e is obtained from $H - e$ and H/e by a $Y\Delta$-exchange. Thus, each of $G - e$ and G/e has $\mathcal{C}$.

This completes the proof. □

By a similar argument, we have the following:

Exercise 3.2. Let $\mathcal{C}$ be a property preserved under taking minors, multiplication of edges, and ΔY-exchanges. Let H be a graph obtained from G by a $Y\Delta$-exchange. Suppose that G does not have $\mathcal{C}$ and suppose that H is a forbidden graph for $\mathcal{C}$. Then G is also a forbidden graph for $\mathcal{C}$.

Proof. (of Theorem 1.6) This follows the Proof of Theorem 1.4. □

Before proving Theorem 1.7, we prove Theorem 1.9.

Proof. (of Theorem 1.9) We construct a primitive embedding of a planar graph joined with two vertices as follows. First, we embed a planar graph G_0 into the 2-sphere S^2 in S^3 so that every loops of G_0 bound open disks in S^2 disjoint from the image of G_0, and that each multi edges of G_0 are mutually parallel in S^2. The 2-sphere S^2 separates S^3 into two 3-balls B^+ and B^-. Let v^+ and v^- be vertices contained in $intB^+$ and $intB^-$ respectively. Next we join v^+ and v^- to each of vertices of G_0 by monotone edges in B^+ and B^- respectively. Then we obtain an embedding ϕ of a planar graph G_0 joined with two vertices v^+ and v^-, say G, and prove that this embedding ϕ is primitive.

If G_0 has a loop, then its image under ϕ bounds an open disk in S^2 disjoint from G_0 by our construction, and hence we may assume that G_0 has no loops by Lemma 2.4. If G_0 has multi edges, then the image of them under ϕ are mutually parallel in S^2 by our construction, and hence we may assume that G_0 has no multi edges by Lemma 2.5.

The proof is done by induction on the number of edges of G_0. When G_0 has no edges, G_0 is a disjoint union of vertices and $\phi(G_0) * (v^+, v^-)$ is primitive since it is a trivial theta-curve. Next, let e be an edge of G_0. Then, $\phi(G - e)$ is primitive by the hypothesis of induction. We shall show that $\phi(G/e)$ is also primitive. Note that $\phi(G/e)$ forms $\phi(G_0/e) * (v^+, v^-)$ with multi edges e_1^+, e_2^+, e_1^-, e_2^- such that $\partial e_i^{\pm} = (\phi(e)/\phi(e)) \cup v^{\pm}$, $e_1^{\pm} \cup e_2^{\pm}$ bounds a disk, whose interior is disjoint from $\phi(G/e)$, coming from the triangle defined by e and $v^{\pm}$. By the hypothesis of induction, $\phi(G_0/e) * (v^+, v^-)$ is primitive, and by Lemma 2.5, $\phi(G/e)$ is primitive. □

The genealogies of the K_7-family and $K_{3,3,1,1}$-family are illustrated in Figure 6 and 8, where "→" means a ΔY-exchange, and all elements of them are in [10] and [9] respectively.

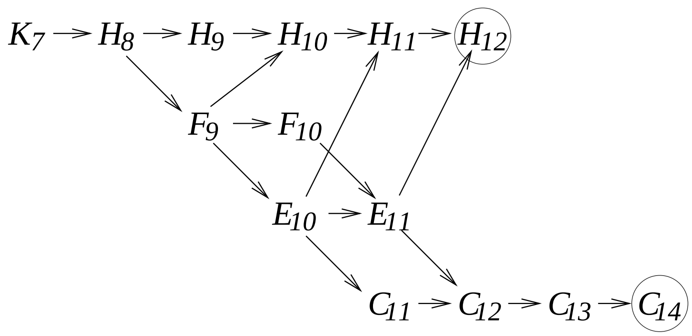

FIGURE 6. K_7-family

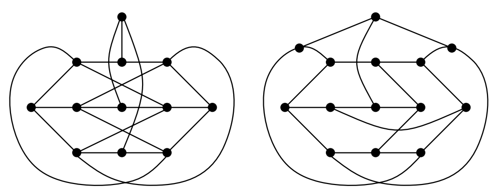

FIGURE 7. H_{12} and C_{14} in the K_7-family

Proof. (of Theorem 1.7) We review that K_7 is intrinsically knotted by Conway-Gordon ([2]) and $K_{3,3,1,1}$ is also intrinsically knotted by Foisy ([3]). Moreover, by Kohara-Suzuki ([10]), the graphs obtained from K_7 or $K_{3,3,1,1}$ by ΔY-exchanges are intrinsically knotted. Thus, K_7-family and $K_{3,3,1,1}$-family are not primitive by Theorem 1.1.

Next, we show that the "terminal" graphs H_{12} and C_{14} in K_7-family and Q_2, Q_3 and R_1 in $K_{3,3,1,1}$-family are forbidden graphs for $\mathcal{P}$. Here, the "terminal" graph means that it can be obtained from K_7 or $K_{3,3,1,1}$ by ΔY-exchanges and does not contain 3-cycles. Thus, K_7-family and $K_{3,3,1,1}$-family are obtained from these terminal graphs by $Y\Delta$-exchanges. Let G be one of these terminal graphs. It can be checked that for any edge e of G, $G - e$ and G/e are planar graphs joined with two vertices. By Theorem 1.9, $G - e$ and G/e are primitive, hence G is a forbidden graph for $\mathcal{P}$.

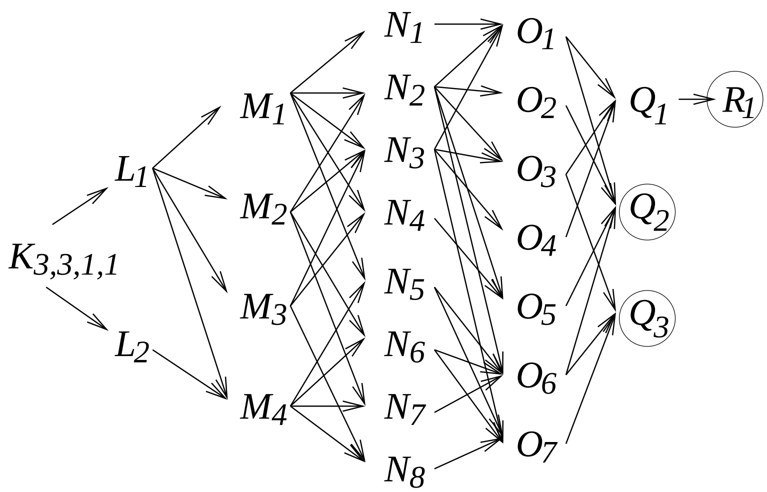

FIGURE 8. $K_{3,3,1,1}$-family

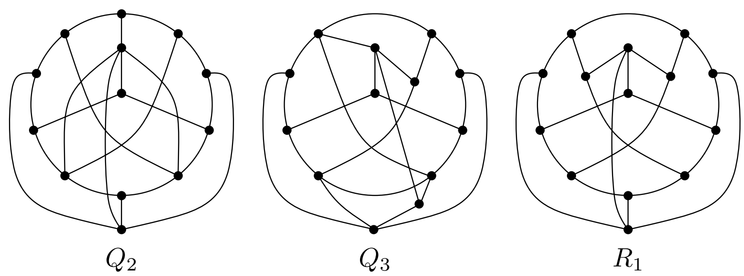

FIGURE 9. Q_2, Q_3 and R_1 in the $K_{3,3,1,1}$-family

We note that $\mathcal{P}$ is preserved under taking minors, multiplication of edges, adding loops, and $Y\Delta$-exchanges. Now, by Lemma 3.1, all graphs in K_7-family and $K_{3,3,1,1}$-family are forbidden graphs for $\mathcal{P}$.

□

4. Proofs of Theorems on primitive embeddings

Proof. (of Theorem 1.11) Let G be a graph without disjoint cycles and ϕ a primitive embedding of G. Then by Theorem 1.1, for any connected subgraph H of G, $\phi(H)$ is free. It is sufficient to show that for any disconnected subgraph $H = H_1 \cup H_2 \cup \cdots \cup H_n$, $\phi(H)$ is also free. Suppose that H_1 contains at least one cycle. Then other connected subgraphs $H_2, \ldots, H_n$ do not contain cycles, so these are trees. Therefore, $\pi_1(S^3 - \phi(H)) \cong \pi_1(\phi(S^3) - \phi(H_1))$, hence $\phi(H)$ is free. □

Proof. (of Theorem 1.12) Put $H_n = C_1 \cup e_1 \cup \cdots \cup e_n \cup C_2$, where C_1 and C_2 are cycles and e_i is an edge connecting C_1 and C_2 for $i = 1, \ldots, n$.

1. Let ϕ be a primitive embedding of H_1. Then by contracting $\phi(e_1)$, we have a primitive embedding $\phi(H_1)/\phi(e_1)$ by Theorem 1.4. Since H_1/e_1 does not contain disjoint cycles, by Theorem 1.11, $\phi(H_1)/\phi(e_1)$ is flat. Hence $\phi(H_1)/\phi(e_1)$ is planar

([18]). By decontracting e_1, we obtain a primitive embedding $\phi(H_1)$ as a 2-bridge link with an upper tunnel. Finally, it is necessary to check that such an embedding is primitive. By Theorem 1.1, it is sufficient to show that any connected subgraph is free, and it is easy to see.

2. Let ϕ be a primitive embedding of H_2. Then by contracting $\phi(e_1)$, we have a primitive embedding $\phi(H_2)/\phi(e_1)$. By Theorem 1.11, $\phi(H_2)/\phi(e_1)$ is flat since H_2/e_1 does not contain disjoint cycles. Hence $\phi(H_2)/\phi(e_1)$ is planar. By decontracting e_1, we obtain a primitive embedding $\phi(H_2)$ as a 2-bridge link with an upper tunnel and a lower tunnel. Finally, it is necessary to check that such an embedding is primitive. By the similar way, it is easy to see.

3. Let ϕ be a primitive embedding of H_3. Then by contracting $\phi(e_1)$, we have a primitive embedding $\phi(H_3)/\phi(e_1)$. By Theorem 1.11, $\phi(H_3)/\phi(e_1)$ is flat since H_3/e_1 does not contain disjoint cycles. Hence $\phi(H_3)/\phi(e_1)$ is planar. By decontracting e_1, we obtain a primitive embedding $\phi(H_3)$ as a 2-bridge link with an upper tunnel and two parallel lower tunnel. But this embedding is not primitive because it may contain a non-trivial knot consisting of e_2, e_3, the paths of length two in C_1 and C_2 (see Figure 10).

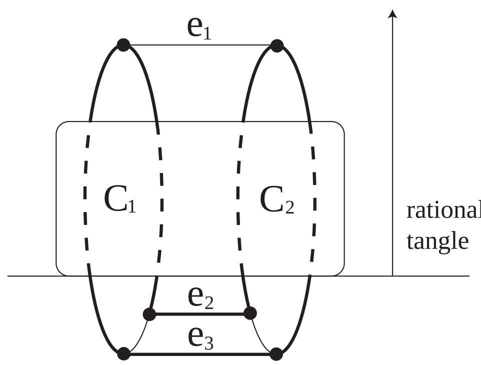

Figure 10. knotted cycle of H_3

A 2-string trivial tangle is usually called a *rational tangle* since it can be represented by a rational number in a standard way. It is necessary and sufficient for the cycle to be unknotted that the continued fraction representation of the upper rational tangle is integral. Thus, the 2-bridge link is a $(2, q)$-torus link and we obtain the desired form. Finally, it is necessary to check that such an embedding is primitive. By the similar way, it is easy to see. □

For any rational tangle (B, T) with fraction r, there exists an orientation preserving homeomorphism of pairs $\phi : (B, T) \to (D^2 \times I, \{x, y\} \times I)$. We call the tangle $\phi^{-1}((D^2 \times I, H))$ a *rational H-shaped tangle with fraction r*, where the tangle $(D^2 \times I, H)$ forms a trivial tangle with an edge. See Figure 11.

Proof. (of Lemma 1.13) Let ϕ be a primitive embedding of K_5'. By Theorem 1.4, $\phi(K_5')/\phi(e)$ is also primitive, and by Theorem 1.11 and (1.7) in [16], primitive embeddings of $K_5'/e \simeq K_5$ are unique up to reflection. Hence, by de-contracting e, we obtain a candidate for primitive embeddings of K_5', where the neighborhood of

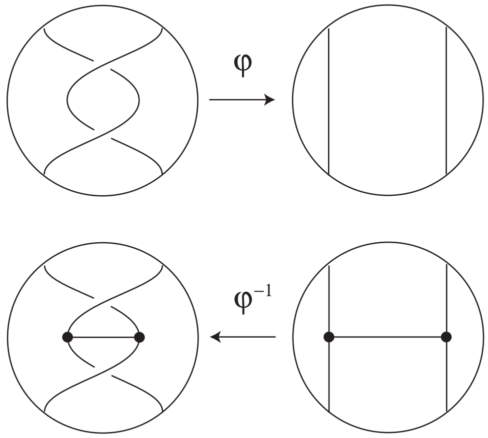

FIGURE 11. a rational H-shaped tangle with fraction 1/2

$\phi(e)$ forms a rational H-shaped tangle, that is, a rational tangle with an additional trivial edge e. In a rational H-shaped tangle, there exists a properly embedded disk that entirely contains the H-shaped part. See Figure 12.

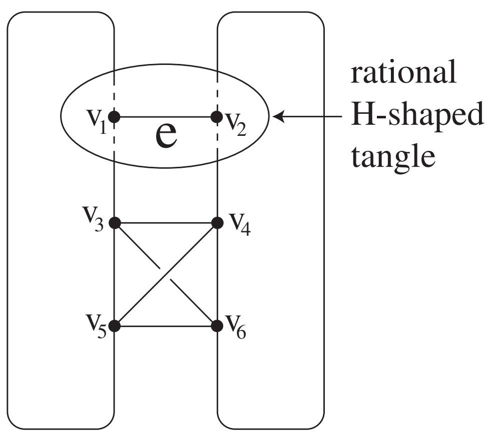

FIGURE 12. a candidate for primitive embeddings of K_5'

By Theorem 1.1, the cycle $v_1v_3v_4v_2v_6v_5v_1$ must be a trivial knot. This implies that the rational tangle has a slope $1/2n$ for some integer n. Moreover, the cycle $v_1v_3v_6v_2v_4v_5v_1$ must be also trivial, it follows that $n = 0$ or $n = -1$. Thus, we conclude that all primitive embeddings of K_5' have only two candidates (a) and (b) in Figure 3 corresponding to $n = 0$ and $n = -1$ respectively. Conversely, by Theorem 1.9, these two embeddings are primitive since they form into a plane graph joined with the North pole and the South pole by adding some vertices if necessary. □

Proof. (of Theorem 1.15) Let G be a graph in the Petersen family, ϕ a primitive embedding of G, and $C_1 \cup C_2$ be a disjoint cycle in G. Then $C_1 \cup C_2$ is contained in a subgraph H of G which has a minor K_5'. Since $\phi(H)$ is primitive, its spatial minor $\phi(K_5')$ is also primitive. Hence by Lemma 1.13, $\phi(C_1 \cup C_2)$ is the trivial link or the Hopf link. □

Proof. (of Theorem 1.16) Let ϕ be a primitive embedding of a graph G, and $C = C_1 \cup \cdots \cup C_n$ a disjoint union of n-cycles of G. First, we contract a path P_i of maximal length in each cycle C_i, and obtain a primitive embedding $\phi(G)/\phi(P_1 \cup \cdots \cup P_n)$ by the proof of Theorem 1.6. We note that by this contraction, the link type of $\phi(C)$ does not change, thus $\phi(C)$ is equivalent to $\phi(C') = \phi(C_1') \cup \cdots \cup \phi(C_n')$, where $\phi(C_i') = \phi(C_i)/\phi(P_i)$. Second, we take a spanning tree T' of the resultant graph $G' = G/(P_1 \cup \cdots \cup P_n)$ and contract $\phi(T')$. Then we obtain a trivial bouquet $\phi(G')/\phi(T')$ by the primitivity of ϕ. In other words, the tangle $(B, \phi(C') \cap B)$ is a trivial n-string tangle, where $B = S^3 - intN(\phi(T'))$. On the other side, the tangle $(N(\phi(T')), \phi(C') \cap N(\phi(T')))$ is also trivial, hence the link $\phi(C')$ is an n-bridge link. As we noted above, the link $\phi(C)$ is also n-bridge. □

Proof. (of Theorem 1.18) Suppose that a planar graph G has no disjoint cycles. Then by Theorem 1.11, any primitive embedding of G is also flat, and by [18], it is planar. Therefore G has a unique primitive embedding.

Conversely, suppose that G has disjoint cycles. Since G is planar, there exists a pair of disjoint facial cycles C_1, C_2. Thus G can be embedded in an annulus A so that $C_1 \cup C_2 = \partial A$. Let $f_n : A \to S^3$ be an embedding such that $f(\partial A)$ forms a $(2, 2n)$-torus link. Then every non-trivial two-component constituent link of $f_n(G)$ also forms a $(2, 2n)$-torus link. In particular, $f_n(C_1 \cup C_2)$ is a $(2, 2n)$-torus link.

Claim 4.1. f_n is a primitive embedding of G for all n.

Proof. Let T be a spanning tree for G. Then there exists a path P in T which connects C_1 and C_2. By contracting $f_n(P)$, we have an embedding $f_n(G)/f_n(P)$ contained in $f_n(A)/f_n(P)$, which is a immersed disk. Then the "twists" of $f_n(A)$ can be untied at the point $f_n(P)/f_n(P)$. Thus $f_n(G)/f_n(P)$ is planar, and then by contracting a rest of edges of $f_n(T)$, we have a trivial bouquet. Hence f_n is primitive for all n. □

Hence G has infinitely many primitive embeddings.

□

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