SATELLITE DOUBLE TORUS KNOTS

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ABSTRACT

We characterize satellite double torus knots. Especially, if a satellite double torus knot is not a cable knot, then it has a torus knot companion. This answers Question 12 (a) raised by Hill and Murasugi in [4].

1. Introduction

A knot $K$ in the 3-sphere $S^3$ is said to be double torus if $K$ is contained in a genus two Heegaard surface $F$ of $S^3$.

A tunnel number one knot $K$ is double torus because $K$ is contained in a genus two Heegaard surface as the boundary of a regular neighborhood of a union of $K$ and an unknotting tunnel for $K$. Morimoto and Sakuma ([7]) characterized satellite tunnel number one knots as follows. Let $K_0$ be a non-trivial torus knot of type $(p, q)$ in $S^3$, and let $L = K_1 \cup K_2$ be a 2-bridge link of type $(\alpha, \beta)$ in $S^3$ with $\alpha \geq 4$ (that is, $L$ is neither a trivial link nor a Hopf link). Since $K_2$ is a trivial knot, there is an orientation preserving homeomorphism $f : E(K_2) \to N(K_0)$ which takes a meridian $m_2 \subset \partial E(K_2)$ of $K_2$ to a fiber $h \subset \partial N(K_0) = \partial E(K_0)$ of the Seifert fibration $D(-r/p, s/q)$ of $E(K_0)$. We denote the knot $f(K_1) \subset N(K_0) \subset S^3$ by the symbol $K(\alpha, \beta; p, q)$. Then the set of satellite tunnel number one knots is the same as the set of all $K(\alpha, \beta; p, q)$. We note that the companion knot of a satellite tunnel number one knot is a torus knot.

A free genus one knot $K$ is double torus because $K$ is contained in a genus two Heegaard surface as the boundary of a regular neighborhood of a genus one free Seifert surface of $K$. In [8], it was shown that a satellite knot bounds a genus one
free Seifert surface if and only if it is \( K(8m, 4m + 1; p, q) \), where \( m \neq 0 \).

In this paper, we characterize satellite double torus knots.

Let \( H \) be an unknotted torus in \( S^3 \), and \( D \) a disk intersecting \( H \) in an arc transversely. Let \( K \) be a knot contained in a twice punctured torus \( H_0 = H - \text{int}N(\partial D) \) such that \( H_0 - K \) is incompressible in \( S^3 - \text{int}N(\partial D) \). Let \( f : S^3 - \text{int}N(\partial D) \to N(K_0) \) be an orientation preserving homeomorphism which takes each boundary component of \( H_0 - K \) is incompressible in \( S^3 - \text{int}N(\partial D) \). Let \( f : S^3 - \text{int}N(\partial D) \to N(K_0) \) be an orientation preserving homeomorphism which takes each boundary component of \( H_0 - K \) is incompressible in \( S^3 - \text{int}N(\partial D) \). Let \( f : S^3 - \text{int}N(\partial D) \to N(K_0) \) be an orientation preserving homeomorphism which takes each boundary component of \( H_0 - K \) is incompressible in \( S^3 - \text{int}N(\partial D) \). Let \( f : S^3 - \text{int}N(\partial D) \to N(K_0) \) be an orientation preserving homeomorphism which takes each boundary component of \( H_0 - K \) is incompressible in \( S^3 - \text{int}N(\partial D) \).

**Theorem 1.** Let \( K \) be a double torus knot in \( S^3 \). Then \( K \) is a satellite knot if and only if it is either

(i) a cable knot of a tunnel number one knot,
(ii) \( K(\alpha, \beta; p, q) \) or
(iii) \( K(H_0, K; p, q) \).

**Remark 1.** These knot classes may contain common knots each other.

2. Preliminaries

A surface \( F \) properly embedded in a 3-manifold \( M \) is essential if it is incompressible and not boundary-parallel in \( M \).

**Lemma 1.** ([9, Lemma 2.3]) Let \( K \) be a double torus knot with respect to a genus two Heegaard splitting \((F; V_1, V_2)\). If \( F - K \) is compressible in \( S^3 - K \), then \( K \) is either a tunnel number one knot or a cable knot of a tunnel number one knot.

**Lemma 2.** ([1, 15.26 Lemma]) Let \( K \) be a knot in \( S^3 \). If \( E(K) \) contains an essential annulus \( A \), then either

(1) \( K \) is a composite knot and \( A \) can be extended to a decomposing sphere for \( K \),
(2) \( K \) is a torus knot and \( A \) can be extended to an unknotted torus or
(3) \( K \) is a cable knot and \( A \) is the cabling annulus.

**Lemma 3.** ([5, Lemma 3.1]) If \( A \) is an incompressible annulus properly embedded in the solid torus \( V \), then \( A \) is boundary-parallel.

Kobayashi characterized essential annuli in a genus two handlebody as follows ([5]).
Lemma 4. ([5, Lemma 3.2]) If $A$ is an essential annulus in a genus two handlebody $W$, then either

(1a) $A$ cuts $W$ into a solid torus $W_1$ and a genus two handlebody $W_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of $W_2$ such that $D_1 \cap A = \emptyset$ and $D_2 \cap A$ is an essential arc in $A$, or

(1b) $A$ cuts $W$ into a genus two handlebody $W'$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of $W'$ such that $D_1 \cap A$ is an essential arc in $A$.

Figure 1: Essential annulus of type (1a)

Lemma 5. ([5, Lemma 3.4]) Let $\{A_1, A_2\}$ be a system of mutually disjoint, non-parallel, essential annuli in the genus two handlebody $W$. Then either

(2a) $A_1 \cup A_2$ cuts $W$ into a solid torus $W_1$ and a genus two handlebody $W_2$. Then $A_1 \cup A_2 \subset \partial W_1$, $A_1 \cup A_2 \subset \partial W_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of $W_2$ such that $D_i \cap A_j = \emptyset$ ($i \neq j$) and $D_i \cap A_i$ ($i = 1, 2$) is an essential arc of $A_i$,

(2b) $A_1 \cup A_2$ cuts $W$ into two solid tori $W_1$, $W_2$ and a genus two handlebody $W_3$. Then $A_1 \subset \partial W_1$, $A_2 \subset W_2$, $A_1 \cup A_2 \subset \partial W_3$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of $W_3$ such that $D_i \cap A_j = \emptyset$ ($i \neq j$) and $D_i \cap A_i$ ($i = 1, 2$) is an essential arc of $A_i$ or

(2c) $A_1 \cup A_2$ cuts $W$ into a solid torus $W_1$ and a genus two handlebody $W_2$. Then $A_1 \subset \partial W_1$ ($i = 1$ or 2, say 1), $A_2 \cap W_1 = \emptyset$, $A_1 \subset \partial W_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of $W_2$ such that $D_1 \cap A_2$ is an essential arc of $A_2$ and $D_2 \cap A_i$ ($i = 1, 2$) is an essential arc of $A_i$. 
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Figure 2: Essential annulus of type (1b)

Figure 3: Essential annuli of type (2a)
Figure 4: Essential annuli of type (2b)

Figure 5: Essential annuli of type (2c)
**Lemma 6.** ([5, Lemma 3.5]) Let \( \{A_1, A_2, A_3\} \) be a system of pairwise disjoint, non-parallel essential annuli in the genus two handlebody \( W \). Then \( A_1 \cup A_2 \cup A_3 \) cuts \( W \) into two solid tori \( W_1, W_2 \) and a genus two handlebody \( W_3 \) which satisfies

\( (3a) \)

1. \( A_i \subset \partial W_1 \) \((i = 1, 2 \text{ or } 3, \text{ say } 3)\), \( A_1, A_2 \subset \partial W_3 \), \( A_1, A_2, A_3 \subset \partial W_2 \).
2. there is a complete system of meridian disks \( \{D_1, D_2\} \) of \( W_3 \) such that \( D_i \cap A_j = \emptyset \) \((i \neq j)\) and \( D_i \cap A_i \) \((i = 1, 2)\) is an essential arc of \( A_i \) and
3. there is a meridian disk \( D_3 \) of \( W_2 \) such that \( D_3 \cap A_i \) \((i = 1, 2, 3)\) is an essential arc of \( A_i \).

![Figure 6: Essential annuli of type (3a)](image)

**Lemma 7.** There exists no system of pairwise disjoint, non-parallel four essential annuli in the genus two handlebody.

**Proof.** Let \( \{A_1, A_2, A_3, A_4\} \) be a system of pairwise disjoint, non-parallel essential annuli in the genus two handlebody \( W \). By Lemma 6, we may assume that \( A_1 \cup A_2 \cup A_3 \) cuts \( W \) into two solid tori \( W_1, W_2 \) and a genus two handlebody \( W_3 \) which satisfies the condition \((3a)\). If \( A_4 \subset W_1 \), then by Lemma 3, \( A_4 \) is parallel to \( A_3 \), a contradiction. Suppose \( A_4 \subset W_2 \). Since by the condition 3 of \((3a)\), \( A_i \) \((i = 1, 2, 3)\) winds around \( W_2 \) exactly once, it follows from Lemma 3 that \( A_4 \) is parallel to one of \( A_1, A_2 \) and \( A_3 \), a contradiction. If \( A_4 \subset W_3 \), then Lemma 5.1 in [6] assures us that \( A_4 \) is parallel to \( A_1 \) or \( A_2 \), a contradiction. \( \square \)
Lemma 8. Let $V$ be a solid torus, $F$ a twice punctured torus properly embedded in $V$ such that each component of $\partial F$ is isotopic to a core of $V$, and $K$ is a knot contained in $F$. Suppose that $F - K$ is incompressible in $V - K$. Then $\partial V$ is incompressible in $V - K$ and $K$ is not isotopic to a core of $V$.

Proof. Suppose $\partial V$ is compressible in $V - K$. Then by compressing $\partial V$ in $V - K$, we obtain an essential sphere $S$ in $V - K$. We take $S$ so that $|S \cap F|$ is minimal up to isotopy of $S$ in $V - K$. Note that $S \cap F \neq \emptyset$ since $K \subset F$ and $F$ separates $\partial F$ and $K$ in $V$. Then an innermost disk in $S$ with respect to $S \cap F$ gives a compressing disk for $F - K$ in $V - K$, a contradiction. Next, suppose $K$ is isotopic to a core of $V$. Then $V - \text{int}N(K)$ is homeomorphic to $(\text{torus}) \times I$ and $\pi_1(V - \text{int}N(K)) \cong \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, since $F - \text{int}N(K)$ is incompressible in $V - \text{int}N(K)$, $\pi_1(F - \text{int}N(K))$ is a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$. This is a contradiction. □

3. Proof of Theorem

Proof. Let $K$ be a satellite double torus knot with respect to a genus two Heegaard splitting $(F; V_1, V_2)$, $T$ an essential torus in $E(K)$. Then $T$ bounds a solid torus $X$ containing $K$. Put $E(X) = S^3 - \text{int}X$.

If $F - K$ is compressible in $S^3 - K$, then by Lemma 1, $K$ is either a tunnel number one knot or a cable knot of a tunnel number one knot. In the former case, by Morimoto and Sakuma’s result, $K = K(\alpha, \beta; p, q)$ and the conclusion (ii) of Theorem 1 holds. In the latter case, we have the conclusion (i) of Theorem 1.

Hereafter, we suppose that $F - K$ is incompressible in $S^3 - K$. We may assume that $T \cap F$ consists of loops, and assume that $|T \cap F|$ is minimal among all essential tori $T$ in $E(K)$. If $T \cap F = \emptyset$, then $T \subset V_i$ and $T$ is compressible in $V_i$. This contradicts the essentiality of $T$. Put $T_i = T \cap V_i$ ($i = 1, 2$).

Claim 1. Each component of $T_i$ is an incompressible annulus in $V_i$.

Proof. Suppose that a component of $T_i$ is compressible. Then by an innermost disk argument, there exists a compressing disk $D$ for some component $P$ of $T_i$ such that $\text{int}D \cap T_i = \emptyset$. Since $T$ is incompressible in $S^3 - K$, $\partial D$ bounds a disk $D'$ in $T$. Note that $|D' \cap F| \geq 1$ since $D$ is a compressing disk for $P$. Then the irreducibility of $S^3 - K$ assures us that there exists an isotopy of $T$ such that $|T \cap F|$ can be reduced. This contradicts the minimality of $|T \cap F|$.

Next, suppose that there exists a component of $T_i$ which is not an annulus. Then there exists a disk component $P$ of $T_1$ or $T_2$, say $T_1$. Since $F - K$ is incompressible in $S^3 - K$, $\partial P$ bounds a disk $P'$ in $F - K$. Then the irreducibility of $V_i$ assures us that $P$ is boundary-parallel in $V_i$ to $P'$. Hence $|T \cap F|$ can be reduced. This contradicts the minimality of $|T \cap F|$. □

Claim 2. Each component of $T_i$ is not $\partial$-parallel in $V_i$. 

Proof. Suppose that there exists a $\partial$-parallel component $P$ of $T_i$, say $T_1$. By exchanging $P$, we may assume that $P$ is outermost in $V_i$, that is, there exists an annulus $P'$ in $F$ to which $P$ is parallel and $\text{int} P' \cap T = \emptyset$. By the minimality of $|T \cap F|$, $P'$ contains $K$ as its core loop. If there exists a $\partial$-parallel component of $T_2$, then by the minimality of $|T \cap F|$, the outermost component of $T_2$ forms $T$ with $P$. Therefore, $K$ is a core loop of $X$ and this contradicts the essentiality of $T$. Otherwise, by Lemmas 4, 5, 6 and 7, some component of $T_2$ is boundary-compressible in $V_2 - K$. This implies that $F - K$ is compressible in $S^3 - K$, a contradiction.

Hence, the parallel class of $(V_i, T_i)$ is either of type (1a), (1b), (2a), (2b), (2c) or (3a). Only one component of $F - T$ is either a twice punctured torus or a 4-punctured sphere, which we denote by $F_K$, and other components are annuli.

Claim 3. $K$ is not parallel to a component of $F \cap T$ in $F$.

Proof. We note that for any type of $(V_i, T_i)$, there exists a compressing disk $D$ for $F_K$ in $V_i$ with $D \cap \partial V_i = \emptyset$. If $K$ is parallel to a component of $F \cap T$ in $F$, then after slight isotopy of $K$, $D$ becomes a compressing disk for $F - K$ in $V_i$, a contradiction. □

Hence $F_K$ contains $K$.

Claim 4. Each component of $F \cap E(X)$ or $(F \cap X) - F_K$ is an essential annulus in $E(X)$ or $X - K$ respectively.

Proof. This follows Claims 1 and 2. □

Claim 5. $E(X)$ is a torus knot exterior.

Proof. By Lemmas 4, 5, 6 and 7, each component of $V_i - T_i$ is either a genus two handlebody or a solid torus. Hence by Lemma 2, $E(X)$ is either a torus knot exterior or a cable knot exterior and each component of $F \cap E(X)$ is the cabling annulus. In the latter case, by cutting and pasting $T$ along the cabling annulus, we obtain a new essential torus $T'$ in $E(K)$ with $|T' \cap F| < |T \cap F|$, a contradiction. □

Hence, $F \cap E(X)$ consists of mutually parallel cabling annuli.

Claim 6. $|F \cap X| = 1$.

Proof. Suppose $|F \cap X| \geq 2$. Since each component of $\partial(F \cap X) = \partial(F \cap E(X))$ winds around $\partial X$ exactly once, each component of $(F \cap X) - F_K$ is boundary-parallel in $X - K$. This contradicts Claim 2. □

Thus $F \cap X = F_K$ and $|F \cap E(X)| = 1$ if $F_K$ is a twice punctured torus, and $|F \cap E(X)| = 2$ if $F_K$ is a 4-punctured sphere.
Claim 7. The parallel class of \((V_i, T_i)\) is neither of type (2c) nor (3a).

**Proof.** If the parallel class of \((V_i, T_i)\) is of type (2c), then \(T_i\) contains at least two mutually parallel non-separating annuli since \(T\) is separating in \(S^3\). However, this contradicts \(|F \cap X| = 1\). If it is of type (3a), then we have \(|F \cap X| > 1\), the same contradiction. □

Claim 8. The parallel class of \((V_i, T_i)\) is not of type (2a).

**Proof.** This follows from that \(F \cap E(X)\) consists of mutually parallel cabling annuli. □

Hence, by observing the loop class and the number of \(T \cap F\) on \(F\), we have the following cases for \((V_1, T_1)\) and \((V_2, T_2)\).

- (1a) – (1a)
- (1b) – (1b)
- (1b) – (2b)
- (2b) – (2b)

Claim 9. The combination (1b) – (1b) does not occur.

**Proof.** There are two parallel classes of \(\partial T_i\) in \(F\), say \(a_1, a_2\) and \(b_1, b_2\). We observe that each component of \(T_i\) is cobounded by \(a_i\) and \(b_i\) with suitable order of suffixes. Hence both \(T_1\) and \(T_2\) have only one component of type (1b), but this does not occur since \(T\) is separating in \(S^3\). □

Claim 10. The combination (2b) – (2b) does not occur.

**Proof.** Otherwise, \(T\) has more than one component. □

Claim 11. By retaking \(F\), we can convert (1b) – (2b) into (1a) – (1a).

**Proof.** We may assume that \((V_1, T_1)\) is of type (1b) and \((V_2, T_2)\) is of type (2b). Since \(|F \cap X| = 1\) and \(|F \cap E(X)| = 2\), \(T_i\) consists of two annuli \((i = 1, 2)\). \(T_1\) cuts \(V_1\) into a genus two handlebody \(W_{11}\) and a solid torus \(W_{12}\) as a product of a component of \(T_1\). \(T_2\) cuts \(V_2\) into two solid tori \(W_{21}\) and \(W_{22}\) and a genus two handlebody \(W_{23}\).

Since each component of \(T \cap F\) winds around \(\partial X\) exactly once, by compressing \(F_K\) along a separating disk \(D\) in \(W_{11}\) with \(D \cap T_1 = \emptyset\), we have two boundary parallel annuli in \(X\). Hence, each component of \(T_1\) winds around a handle of \(V_1\) exactly once. Therefore, if we attach a solid torus \(W_{21}\) to \(V_1\), then we obtain a genus two handlebody again. On the other side, if we remove \(W_{21}\) from \(V_2\), then we
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have a genus two handlebody. Hence we have a new genus two Heegaard splitting $(F'; V_1 \cup W_{21}, V_2 - W_{21})$ with $F' \supset K$, and after a slight isotopy of $F'$, we have a configuration of type (1a)-(1a).

Hence, we may conclude that the case for $(V_1, T_1)$ and $(V_2, T_2)$ is (1a)-(1a), and that $T_i$ consists of a single annulus ($i = 1, 2$).

If we attach a 2-handle $H_i$ to $V_i \cap X$ along $T_i$, then we get a solid torus since there exists a disk $D_i$ such that $D_i \cap T_i$ is an essential arc in $T_i$ ($i = 1, 2$). Since $\partial T_i$ winds around $X$ exactly once, $X \cup (H_1 \cup H_2)$ is the 3-sphere, and the core loop $J$ of a solid torus $H_1 \cup H_2$ is a trivial one bridge knot with respect to the genus one Heegaard splitting $((V_1 \cap X) \cup H_1) \cup ((V_2 \cap X) \cup H_2)$. By Theorem B and Lemma 2.2 in [2], $J$ bounds a disk $D$ in $X \cup (H_1 \cup H_2)$ which intersects the genus one Heegaard surface $\partial((V_1 \cap X) \cup H_1)$ in an arc. Finally, since $F - K$ is incompressible in $S^3 - K$, $F_K - K$ is also incompressible in $X - K$. Thus we have a conclusion (ii) of Theorem 1.

Conversely, if $K$ is a cable knot of a tunnel number one knot $K'$, then $K$ can be isotoped so that it lies on the boundary of a regular neighborhood of $K'$ and an unknotting tunnel for $K'$ naturally. Thus $K$ is a satellite double torus knot. If $K$ is $K(H_0, K; p, q)$, then $K$ is contained in a union of $f(H_0)$ and a unique essential annulus in a torus knot exterior $E(K_0)$. Moreover, Lemma 8 assures us that $K$ is a satellite double torus knot. This completes the proof of Theorem 1.

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